

Perturbations of Matter Fields in the Second-order Gauge-invariant Cosmological Perturbation Theory

Kouji NAKAMURA

Department of Astronomical Science, the Graduate University for Advanced Studies, Mitaka, Tokyo 181-8588, Japan.

Some formulae for the perturbations of the matter fields are summarized within the framework of the second-order gauge-invariant cosmological perturbation theory in a four dimensional homogeneous isotropic universe, which is developed in the papers [K. Nakamura, Prog. Theor. Phys. **117** (2007), 17.]. We derive the formulae for the perturbations of the energy momentum tensors and equations of motion for a perfect fluid, an imperfect fluid, and a single scalar field, and show that all equations are derived in terms of gauge-invariant variables without any gauge fixing.

§1. Introduction

The general relativistic second-order cosmological perturbation theory is one of topical subjects in the recent cosmology. By the recent observation of the Cosmic Microwave Background (CMB) by Wilkinson Microwave Anisotropy Probe,¹⁾ the first order approximation of our universe from a homogeneous isotropic one was revealed. The observational results suggest that the fluctuations of our universe are adiabatic and Gaussian at least in the first order approximation. As a next step, the clarifications of the accuracy of these observational results are actively discussed both in the observational²⁾ and the theoretical side^{3),4)} through the non-Gaussianity, the non-adiabaticity, and so on. With the increase of precision of the CMB data, the study of relativistic cosmological perturbations beyond linear order is a topical subject especially to study the generation of the primordial non-Gaussianity in inflationary scenarios³⁾ and the non-Gaussian component in CMB anisotropy.⁴⁾ The second-order cosmological perturbation theory is one of such perturbation theories beyond linear order.

According to this physical motivation, we proposed a clear gauge-invariant formulation of the second-order general relativistic cosmological perturbation theory.⁵⁾ In this paper, we refer these works as KN2007. This gauge-invariant formulation of the second-order cosmological perturbations is a natural extension of the first-order gauge-invariant cosmological perturbation theory.^{6)–8)} The formulation in KN2007 is one of the applications of the gauge-invariant formulation of the second-order perturbation theory on the generic background spacetime developed in the papers by the present author.^{9),10)} These papers are referred in this paper as KN2003⁹⁾ and KN2005.¹⁰⁾ This general formulation is a by-product of the investigations of the oscillatory behaviors of self-gravitating Nambu-Goto membranes.¹¹⁾

In KN2007, we defined the complete set of the gauge-invariant variables of the second-order cosmological perturbations in the Friedmann-Robertson-Walker universe based on the formulation developed in the papers KN2003 and KN2005. We

considered the two cases of the Friedmann-Robertson-Walker universe: one is the universe filled with the single perfect fluid and another is the universe filled with the single scalar field. We also derived the second-order Einstein equations of cosmological perturbations in terms of these gauge-invariant variables without any gauge fixing in these two cases. We have also found that the procedure to find gauge invariant variables proposed in KN2003 plays a crucial role in the derivations. Rather, we can use the formulae proposed in KN2003 to check whether the resulting formulae are correct or not.

This paper is the second part of KN2007. In this paper, we summarize the formulae for the components of the first- and the second-order perturbations of the energy momentum tensors and the equations of motion which are derived from the divergence of the energy momentum tensors. As the matter contents, we consider the three matter fields: a single perfect fluid; a single imperfect fluid which includes additional terms of the energy flux and anisotropic stress to the perfect fluid; and a single scalar field. In the early universe, photon and neutrinos should be described by the Boltzmann distribution functions.^{7), 14)} Photon's interaction with baryon and the free streaming of neutrinos lead anisotropic stress and these effects will be important in the recent cosmology. Although the energy flux and the anisotropic stress in the imperfect fluid are determined through these micro-physical processes, we just phenomenologically treat these terms in this paper.

Although the perturbative expressions of the energy momentum tensor and equations of motion were also derived in some literatures,¹⁶⁾ in this paper, we show alternative derivations of these perturbations. In our derivations, we respect the gauge invariance of the perturbative variables. We again show that the formulae of gauge invariant-variables proposed in KN2003 [Eqs. (2·20) and (2·21) in this paper] also play crucial roles in the derivations of perturbative expressions of the equations for matter fields. The first- and the second-order perturbations of the equations of motion are decomposed into gauge-invariant and gauge-variant parts as Eqs. (2·20) and (2·21), respectively. In these derivations, we do not fix any gauge degree of freedom. In spite of this no gauge-fixing, we show all perturbations of the equations of motion are given in gauge-invariant forms through the lower order perturbations of the equations of motion for matter fields. In this sense, we may say that the general framework of the second-order gauge invariant perturbations proposed in KN2003 and KN2005 does work not only in the perturbations of the Einstein equations but also in the equations of motion for the matter fields. The main purpose of this paper is to show this.

Further, in this paper, we do not ignore the first-order vector- and tensor-modes which are ignored in KN2007. Moreover, in the derivation of the perturbations of the energy momentum tensors and the equations of motion, we do not use any information of the Einstein equations. Therefore, the formulae derived in this paper are valid even if we consider any other theory of gravity than the Einstein theory.

The organization of this paper is as follows. In §2, we briefly review the definitions of the gauge-invariant variables for the first- and second-order perturbations which were defined by KN2007.⁵⁾ In §3, we derive the first- and the second-order perturbations of the energy momentum tensors and equations of motion for a perfect

fluid, an imperfect fluid, and a scalar field. In the derivation in this section, we do not specify the background spacetime. Therefore, the formulae summarized in this section are valid in perturbation theories on any background spacetime. In §4, we derive the explicit expression of the components of the energy momentum tensors and the equations of motion for matter fields. The final section, §5, is devoted to the summary and the discussions. Further, in Appendix A, we explicitly give the components of the perturbations of the acceleration, expansion, shear, and rotation associated with the fluid four-velocity, which are necessary to derive the results in §§3 and 4.

We employ the notation of our previous papers KN2003,⁹⁾ KN2005,¹⁰⁾ and KN2007⁵⁾ and use the abstract index notation.¹⁷⁾ We also employ the natural unit in which the light velocity is denoted by $c = 1$.

§2. Gauge-invariant variables in the second-order perturbations

In this section, we briefly review the definitions of the gauge-invariant variables. First, in §2.1, we review the gauge degree of freedom and the first- and the second-order gauge transformation rules. Then, we briefly explain the gauge-invariant variables for the metric (in §2.2) and matter perturbations (in §2.3).

2.1. Gauge degree of freedom

In any perturbation theory, we always treat two spacetime manifolds. One is the physical spacetime $\mathcal{M} = \mathcal{M}_\lambda$ and the other is the background spacetime \mathcal{M}_0 . The physical spacetime \mathcal{M}_λ is our nature itself which we want to describe through the perturbations. On the other hand, the background spacetime \mathcal{M}_0 is just a reference spacetime for the calculations of perturbations. Although this background spacetime has nothing to do with our nature, to calculate perturbations, it is necessary to introduce this reference spacetime \mathcal{M}_0 by hand. Since these two spacetime manifolds are distinct from each other, we have to introduce a point-identification map $\mathcal{X}_\lambda : \mathcal{M}_0 \rightarrow \mathcal{M}_\lambda$. This point-identification map \mathcal{X}_λ called a gauge choice in perturbation theories. Through the pull-back \mathcal{X}_λ^* of the gauge choice \mathcal{X}_λ , any physical variable \bar{Q}_λ on the physical manifold \mathcal{M}_λ is pulled back to $\mathcal{X}_\lambda^* \bar{Q}_\lambda$ on the background spacetime \mathcal{M}_0 . The pull-back $\mathcal{X}_\lambda^* \bar{Q}_\lambda$ is a representation on the background spacetime \mathcal{M}_0 of the physical variable \bar{Q}_λ on the physical spacetime \mathcal{M}_λ . Although we do not know about the physical spacetime \mathcal{M}_λ at the starting point of the perturbation theory, we can treat the physical variable \bar{Q}_λ on the physical manifold \mathcal{M}_λ as the variable $\mathcal{X}_\lambda^* \bar{Q}_\lambda$ on the background spacetime \mathcal{M}_0 through this pull-back \mathcal{X}_λ^* .

In the case of the perturbations in the theory with general covariance, the above strategy of the perturbation theory includes an important trouble. This is the fact that the gauge choice \mathcal{X}_λ is not unique by virtue of general covariance. Rather, there is degree of freedom in the gauge choice \mathcal{X}_λ , i.e., we may apply the different point-identification map \mathcal{Y}_λ from \mathcal{X}_λ as a gauge choice. In this case, the representation $\mathcal{Y}_\lambda^* \bar{Q}_\lambda$ on the background spacetime \mathcal{M}_0 of the physical variable \bar{Q}_λ on the physical spacetime \mathcal{M}_λ is different from the representation $\mathcal{X}_\lambda^* \bar{Q}_\lambda$. This difference is unphysical because it has nothing to do with the nature of the physical spacetime

\mathcal{M}_λ . We can also consider the transformation rule from a gauge choice \mathcal{X}_λ to another one \mathcal{Y}_λ , which is called gauge transformation. The gauge transformation $\mathcal{X}_\lambda \rightarrow \mathcal{Y}_\lambda$ is induced by the diffeomorphism $\Phi_\lambda := (\mathcal{X}_\lambda)^{-1} \circ \mathcal{Y}_\lambda$. Actually, the diffeomorphism Φ_λ changes the point-identification maps from \mathcal{X}_λ to \mathcal{Y}_λ . Further, the pull-back Φ_λ^* of the diffeomorphism Φ_λ changes the representation $\mathcal{X}_\lambda^* \bar{Q}_\lambda$ of the physical variable \bar{Q}_λ to another representation $\mathcal{Y}_\lambda^* \bar{Q}_\lambda$:

$$\mathcal{Y}_\lambda^* \bar{Q}_\lambda = \mathcal{Y}_\lambda^* (\mathcal{X}_\lambda \circ \mathcal{X}_\lambda^{-1})^* \bar{Q}_\lambda = \mathcal{Y}_\lambda^* (\mathcal{X}_\lambda^{-1})^* \mathcal{X}_\lambda^* \bar{Q}_\lambda = \Phi_\lambda^* \mathcal{X}_\lambda^* \bar{Q}_\lambda. \quad (2.1)$$

Since Φ_λ is the diffeomorphism on the background manifold \mathcal{M}_0 , the Taylor expansion of the pull-back Φ_λ^* is given by

$$\Phi_\lambda^* \mathcal{X}_\lambda^* \bar{Q}_\lambda = \mathcal{X}_\lambda^* \bar{Q}_\lambda + \lambda \mathcal{L}_{\xi_1} \mathcal{X}_\lambda^* \bar{Q}_\lambda + \frac{1}{2} \lambda^2 (\mathcal{L}_{\xi_2} + \mathcal{L}_{\xi_1}^2) \mathcal{X}_\lambda^* \bar{Q}_\lambda + O(\lambda^3), \quad (2.2)$$

where ξ_1^a and ξ_2^a are generators of the diffeomorphism Φ_λ^* .¹³⁾ On the other hand, we consider the perturbative expansion

$$\mathcal{X}_\lambda^* \bar{Q}_\lambda = Q_0 + \lambda \mathcal{X}^{(1)} Q + \frac{1}{2} \lambda^2 \mathcal{X}^{(2)} Q + O(\lambda^3) \quad (2.3)$$

of the representation $\mathcal{X}_\lambda^* \bar{Q}_\lambda$ under the gauge choice \mathcal{X}_λ , where Q_0 is the background value of the variable \bar{Q}_λ . The first- and the second-order perturbations $\mathcal{X}^{(1)} Q$ and $\mathcal{X}^{(2)} Q$ are defined by this equation (2.3). From Eqs. (2.2) and (2.3), we can easily derive the gauge transformation rule of each order:

$$\mathcal{Y}^{(1)} Q - \mathcal{X}^{(1)} Q = \mathcal{L}_{\xi_{(1)}} Q_0, \quad (2.4)$$

$$\mathcal{Y}^{(2)} Q - \mathcal{X}^{(2)} Q = 2 \mathcal{L}_{\xi_{(1)}} \mathcal{X}^{(1)} Q + \left\{ \mathcal{L}_{\xi_{(2)}} + \mathcal{L}_{\xi_{(1)}}^2 \right\} Q_0. \quad (2.5)$$

Further, we introduce the concept of “*the order by order gauge invariance*”. We call the p th-order perturbation $\mathcal{X}^{(p)} Q$ is gauge invariant iff

$$\mathcal{Y}^{(p)} Q = \mathcal{X}^{(p)} Q \quad (2.6)$$

for any gauge choice \mathcal{X}_λ and \mathcal{Y}_λ . We have been considering the concept of “*the gauge invariant up to order n* ” in the series of the papers KN2003,⁹⁾ KN2005,¹⁰⁾ and KN2007,⁵⁾ following the idea by Bruni and Sonogo.¹²⁾ However, we should regard the gauge invariance in this series of the papers is this “order by order gauge invariance” rather than “the gauge invariance up to order n ”. This notion of the order by order gauge invariance is weaker than the notion of the gauge invariance up to n , since we do not say anything about the gauge invariance of the other orders in the above order by order gauge invariance.

Employing the idea of this order by order gauge invariance, we proposed a procedure to construct gauge-invariant variables of higher-order perturbations in KN2003.⁹⁾ Inspecting the gauge transformation rules (2.4) and (2.5), we can define the gauge-invariant variables for a metric perturbation and for arbitrary matter fields.

2.2. Gauge-invariant variables for metric perturbations

Following the expansion form (2.3), we also expand the pulled-back $\mathcal{X}_\lambda^* \bar{g}_{ab}$ of the metric \bar{g}_{ab} on the physical spacetime \mathcal{M}_λ :

$$\mathcal{X}_\lambda^* \bar{g}_{ab} = g_{ab} + \lambda \mathcal{X} h_{ab} + \frac{\lambda^2}{2} \mathcal{X} l_{ab} + O(\lambda^3), \quad (2.7)$$

where g_{ab} is the metric on the background spacetime \mathcal{M}_0 . Although the expansion (2.7) of the metric depends entirely on the gauge choice \mathcal{X}_λ , henceforth, we do not explicitly express the index of the gauge choice \mathcal{X}_λ in expressions if there is no possibility of confusion. As shown in KN2007,⁵⁾ at least in the cosmological perturbation case, the first-order metric perturbation h_{ab} is decomposed as

$$h_{ab} =: \mathcal{H}_{ab} + \mathcal{L}_X g_{ab}, \quad (2.8)$$

where \mathcal{H}_{ab} and X^a are the gauge-invariant and gauge-variant parts of the linear-order metric perturbations,⁹⁾ i.e., under the gauge transformation (2.4), these are transformed as

$$\mathcal{Y} \mathcal{H}_{ab} - \mathcal{X} \mathcal{H}_{ab} = 0, \quad \mathcal{Y} X^a - \mathcal{X} X^a = \xi_{(1)}^a. \quad (2.9)$$

Further, the second-order metric perturbation l_{ab} is also decomposed as

$$l_{ab} =: \mathcal{L}_{ab} + 2\mathcal{L}_X h_{ab} + (\mathcal{L}_Y - \mathcal{L}_X^2) g_{ab}, \quad (2.10)$$

where \mathcal{L}_{ab} and Y^a are the gauge-invariant and gauge-variant parts of the second-order metric perturbations, i.e., these are transformed as

$$\mathcal{Y} \mathcal{L}_{ab} - \mathcal{X} \mathcal{L}_{ab} = 0, \quad \mathcal{Y} Y^a - \mathcal{X} Y^a = \xi_{(2)}^a + [\xi_{(1)}, X]^a \quad (2.11)$$

under the gauge transformation (2.5).

In KN2007,⁵⁾ the details of the derivation of this gauge-invariant part of the second-order metric perturbation are explained in the context of cosmological perturbations. In the case of the cosmological perturbations, we consider the homogeneous isotropic background spacetime whose metric is given by

$$g_{ab} = a^2 \left\{ -(d\eta)_a (d\eta)_b + \gamma_{ij} (dx^i)_a (dx^j)_b \right\}, \quad (2.12)$$

where $\gamma_{ab} := \gamma_{ij} (dx^i)_a (dx^j)_b$ is the metric on the maximally symmetric three space. As shown in KN2007, the decomposition (2.8) is accomplished if we assume the existence of the Green functions $\Delta^{-1} := (D^i D_i)^{-1}$, $(\Delta + 2K)^{-1}$, and $(\Delta + 3K)^{-1}$, where D_i is the covariant derivative associated with the metric γ_{ij} on the maximally symmetric three space and K is the curvature constant of this maximally symmetric three space. We also showed in KN2007 that we may choose the components of the gauge-invariant part \mathcal{H}_{ab} of the first-order metric perturbation as

$$\begin{aligned} \mathcal{H}_{ab} = a^2 \left\{ -2 \overset{(1)}{\Phi} (d\eta)_a (d\eta)_b + 2 \overset{(1)}{\nu}_i (d\eta)_{(a} (dx^i)_{b)} \right. \\ \left. + \left(-2 \overset{(1)}{\Psi} \gamma_{ij} + \overset{(1)}{\chi}_{ij} \right) (dx^i)_a (dx^j)_b \right\}, \end{aligned} \quad (2.13)$$

where $\nu_i^{(1)}$ and $\chi_{ij}^{(1)}$ satisfy the properties

$$D^i \nu_i^{(1)} := \gamma^{ij} D_i \nu_j^{(1)} = 0, \quad \chi_i^i := \gamma^{ij} \chi_{ij}^{(1)} = 0, \quad D^i \chi_{ij}^{(1)} = 0. \quad (2.14)$$

Further, we may also choose the components of the gauge-invariant part \mathcal{L}_{ab} of the second-order metric perturbation as

$$\begin{aligned} \mathcal{L}_{ab} = a^2 \bigg\{ & -2 \Phi^{(2)} (d\eta)_a (d\eta)_b + 2 \nu_i^{(2)} (d\eta)_{(a} (dx^i)_{b)} \\ & + \left(-2 \Psi^{(2)} \gamma_{ij} + \chi_{ij}^{(2)} \right) (dx^i)_a (dx^j)_b \bigg\}, \end{aligned} \quad (2.15)$$

where $\nu_i^{(2)}$ and $\chi_{ij}^{(2)}$ satisfy the properties

$$D^i \nu_i^{(2)} = 0, \quad \chi_i^i = 0, \quad D^i \chi_{ij}^{(2)} = 0. \quad (2.16)$$

Here, we also note the fact that the definitions (2.8) and (2.10) of the gauge-invariant variables are not unique. This comes from the fact that we can always construct new gauge-invariant quantities by the combination of the gauge-invariant variables. For example, using the gauge-invariant variables $\Phi^{(1)}$ and $\nu_i^{(1)}$ of the first-order metric perturbation, we can define a vector field Z_a by

$$Z_a := a \Phi^{(1)} (d\eta)_a + a \nu_i^{(1)} (dx^i)_a. \quad (2.17)$$

Although there is no specific physical meaning in this vector field Z_a , at least, we can say that the vector field Z_a defined by (2.17) is gauge-invariant. Through this gauge-invariant vector field Z_a , we can rewrite the decomposition formula (2.8) for the linear-order metric perturbation as

$$\begin{aligned} h_{ab} &= \mathcal{H}_{ab} - \mathcal{L} Z g_{ab} + \mathcal{L} Z g_{ab} + \mathcal{L} X g_{ab}, \\ &=: \mathcal{K}_{ab} + \mathcal{L} X + Z g_{ab}, \end{aligned} \quad (2.18)$$

where we have defined new gauge-invariant variable \mathcal{K}_{ab} by

$$\mathcal{K}_{ab} := \mathcal{H}_{ab} - \mathcal{L} Z g_{ab}. \quad (2.19)$$

Clearly, \mathcal{K}_{ab} is gauge-invariant and the vector field $X^a + Z^a$ satisfies the gauge transformation rule (2.9) for the gauge-variant part of the first-order metric perturbations. Although the definition of the gauge-invariant variables is not unique, we can specify the components of the gauge-variant part X_a without ambiguities if we specify the components of the gauge-invariant part \mathcal{H}_{ab} as shown in KN2007. In this paper, we specify the components of the tensor \mathcal{H}_{ab} as Eq. (2.13), which is the gauge-invariant part of the linear-order metric perturbation associated with the longitudinal gauge.

2.3. Gauge-invariant variables for matter fields

As shown in KN2003, using the above first- and second-order gauge-variant parts, X^a and Y^a , of the metric perturbations, we can define the gauge-invariant variables for an arbitrary field Q other than the metric. These definitions imply that the first- and the second-order perturbations $^{(1)}Q$ and $^{(2)}Q$ are always decomposed into gauge-invariant part and gauge-variant part as

$$^{(1)}Q =: ^{(1)}\mathcal{Q} + \mathcal{L}_X Q_0, \quad (2.20)$$

$$^{(2)}Q =: ^{(2)}\mathcal{Q} + 2\mathcal{L}_X ^{(1)}Q + \{\mathcal{L}_Y - \mathcal{L}_X^2\} Q_0, \quad (2.21)$$

respectively, where $^{(1)}\mathcal{Q}$ and $^{(2)}\mathcal{Q}$ are gauge-invariant parts of the first- and the second-order perturbations of $^{(1)}Q$ and $^{(2)}Q$, respectively.

Through the formulae (2.20) and (2.21), we can define the gauge-invariant variables for the matter field. In this paper, we consider the cases of a perfect fluid; an imperfect fluid; and a scalar field. All these matter fields consist of fundamental quantities. For example, we regard the energy density, the pressure, and the four-velocity as fundamental variables for a perfect fluid. Here, we show the definitions of the gauge-invariant variables for these fundamental quantities. These definitions are just following to the formulae (2.20) and (2.21). However, based on these definitions of gauge-invariant variables, we will show that all perturbative quantities are decomposed into gauge-invariant and gauge-variant parts as Eqs. (2.20) and (2.21).

2.3.1. Perfect fluid

Here, we consider the definitions of the gauge-invariant variables for the fundamental variables of a perfect fluid. The total energy momentum tensor of the fluid is given by

$$^{(p)}\bar{T}_a{}^b = (\bar{\epsilon} + \bar{p})\bar{u}_a\bar{u}^b + \bar{p}\delta_a{}^b, \quad (2.22)$$

where the fundamental variables for a perfect fluid are the energy density $\bar{\epsilon}$, the pressure \bar{p} , and the four-velocity \bar{u}^a . We expand these fundamental variables as

$$\bar{\epsilon} := \epsilon + \lambda \epsilon^{(1)} + \frac{1}{2}\lambda^2 \epsilon^{(2)} + O(\lambda^3), \quad (2.23)$$

$$\bar{p} := p + \lambda p^{(1)} + \frac{1}{2}\lambda^2 p^{(2)} + O(\lambda^3), \quad (2.24)$$

$$\bar{u}_a := u_a + \lambda u_a^{(1)} + \frac{1}{2}\lambda^2 u_a^{(2)} + O(\lambda^3), \quad (2.25)$$

where $\bar{\epsilon}$, \bar{p} , and \bar{u}_a characterize the pull-back of the fluid on the physical spacetime to the background spacetime through an appropriate gauge choice \mathcal{X}_λ , while ϵ , p , and u_a are their background values on the background spacetime. Following to Eqs. (2.20) and (2.21), we define the gauge-invariant variable for the perturbation of the fluid components $\bar{\epsilon}$, \bar{p} , and \bar{u}_a :

$$\mathcal{E} := \epsilon^{(1)} - \mathcal{L}_X \epsilon, \quad \mathcal{P} := p^{(1)} - \mathcal{L}_X p, \quad \mathcal{U}_a := u_a^{(1)} - \mathcal{L}_X u_a, \quad (2.26)$$

$$\mathcal{E} := \epsilon^{(2)} - 2\mathcal{L}_X \epsilon^{(1)} - \{\mathcal{L}_Y - \mathcal{L}_X^2\} \epsilon, \quad \mathcal{P} := p^{(2)} - 2\mathcal{L}_X p^{(1)} - \{\mathcal{L}_Y - \mathcal{L}_X^2\} p, \quad (2.27)$$

$$\mathcal{U}_a := (u_a)^{(2)} - 2\mathcal{L}_X (u_a)^{(1)} - \{\mathcal{L}_Y - \mathcal{L}_X^2\} u_a. \quad (2.28)$$

2.3.2. Imperfect fluid

Here, we consider the generic case of an imperfect fluid. The energy-momentum tensor is decomposed into fluid quantities based on the orthogonality to the four-vector field \bar{u}^a as

$$\bar{T}_a{}^b = \bar{\epsilon} \bar{u}_a \bar{u}^b + \bar{p} \left(\delta_a{}^b + \bar{u}_a \bar{u}^b \right) + \bar{q}_a \bar{u}^b + \bar{u}_a \bar{q}^b + \bar{\pi}_a{}^b \quad (2.29)$$

$$= {}^{(p)}\bar{T}_a{}^b + \bar{g}^{bc} \bar{q}_a \bar{u}_c + \bar{g}^{bc} \bar{u}_a \bar{q}_c + \bar{g}^{bc} \bar{\pi}_{ac}, \quad (2.30)$$

where

$$\bar{u}^a \bar{q}_a = 0, \quad (2.31)$$

$$\bar{\pi}_{[ab]} = 0, \quad \bar{u}^a \bar{\pi}_{ab} = 0, \quad \bar{\pi}_a{}^a = \bar{g}^{ab} \bar{\pi}_{ab} = 0. \quad (2.32)$$

The energy density $\bar{\epsilon}$, the isotropic pressure \bar{p} , and the four-velocity \bar{u}_a of the imperfect fluid in Eq. (2.29) are expanded as Eqs. (2.23)–(2.25) and the gauge-invariant variables for their perturbations are defined by Eqs. (2.26)–(2.28) as in §2.3.1. In addition to these fluid components, in the imperfect fluid case, we add the energy flux \bar{q}_a and the anisotropic stress $\bar{\pi}_{ab}$ associated with the vector field \bar{u}_a as fluid components. Since the first two terms in Eq. (2.29) coincide with the energy momentum tensor for a perfect fluid, we call these terms as “*the perfect part*” and denote them by ${}^{(p)}\bar{T}_a{}^b$ as in Eq. (2.30). On the other hand, we call the remaining terms in Eq. (2.30)

$${}^{(i)}\bar{T}_a{}^b := \bar{q}_a \bar{u}^b + \bar{u}_a \bar{q}^b + \bar{\pi}_a{}^b, \quad (2.33)$$

as “*the imperfect part*” of the energy momentum tensor for an imperfect fluid.

Now, we consider the perturbative expansion of the energy flux \bar{q}_a and the anisotropic stress $\bar{\pi}_{ab}$. Although these quantities should be given through the micro-physical process, in this paper, we regard these variables as fundamental quantities for an imperfect fluid and expand these variables as

$$\bar{q}_a =: q_a + \lambda (q_a)^{(1)} + \frac{1}{2} \lambda^2 (q_a)^{(2)} + O(\lambda^3), \quad (2.34)$$

$$\bar{\pi}_{ab} =: \pi_{ab} + \lambda (\pi_{ab})^{(1)} + \frac{1}{2} \lambda^2 (\pi_{ab})^{(2)} + O(\lambda^3). \quad (2.35)$$

Further, we introduce gauge-invariant variables for the perturbations of the energy flux q_a and the anisotropic stress π_{ab} . Following to the decompositions (2.20) and (2.21) for an arbitrary matter field, the first- and the second-order perturbations of the energy flux and the anisotropic stress are decomposed into gauge-invariant and gauge-variant parts as

$$(q_a)^{(1)} =: \mathcal{Q}_a + \mathcal{L}_X q_a, \quad (q_a)^{(2)} =: \mathcal{Q}_a + 2\mathcal{L}_X (q_a)^{(1)} + \{\mathcal{L}_Y - \mathcal{L}_X^2\} q_a, \quad (2.36)$$

$$(\pi_{ab})^{(1)} =: \Pi_{ab} + \mathcal{L}_X \pi_{ab}, \quad (\pi_{ab})^{(2)} =: \Pi_{ab} + 2\mathcal{L}_X (\pi_{ab})^{(1)} + \{\mathcal{L}_Y - \mathcal{L}_X^2\} \pi_{ab}. \quad (2.37)$$

If we represent the multi-fluid system as an imperfect fluid system or we consider the micro-physical process, these energy flux and the anisotropic stress are related to the other fluid components.^{7),14)} In this case, the gauge transformations (2.36) and (2.37) should be derived from this relation.¹⁵⁾ However, in this paper, we regard these energy flux and the anisotropic stress as fundamental variables for an imperfect fluid, phenomenologically.

2.3.3. Scalar field

Finally, we briefly summarize the perturbations of a scalar field $\bar{\varphi}$ whose energy momentum tensor is given by

$$\bar{T}_a{}^b = \bar{g}^{bc} \bar{\nabla}_a \bar{\varphi} \bar{\nabla}_c \bar{\varphi} - \frac{1}{2} \delta_a{}^b \left(\bar{g}^{cd} \bar{\nabla}_c \bar{\varphi} \bar{\nabla}_d \bar{\varphi} + 2V(\bar{\varphi}) \right), \quad (2.38)$$

where $V(\bar{\varphi})$ is the potential of the scalar field $\bar{\varphi}$. The fundamental variable of this system is the scalar field $\bar{\varphi}$ itself. In perturbation theory, we also expand this scalar field $\bar{\varphi}$ as

$$\bar{\varphi} = \varphi + \lambda \hat{\varphi}_1 + \frac{1}{2} \lambda^2 \hat{\varphi}_2 + O(\lambda^3), \quad (2.39)$$

where φ is the background value of the scalar field $\bar{\varphi}$. Further, as in the cases of the fluids, each order perturbations of the scalar field φ is decomposed into the gauge-invariant part and gauge-variant part as

$$\hat{\varphi}_1 =: \varphi_1 + \mathcal{L}_X \varphi, \quad (2.40)$$

$$\hat{\varphi}_2 =: \varphi_2 + 2\mathcal{L}_X \hat{\varphi}_1 + (\mathcal{L}_Y - \mathcal{L}_X^2) \varphi, \quad (2.41)$$

where φ_1 and φ_2 are the first-order and the second-order gauge-invariant perturbation of the scalar field.

§3. Generic form of perturbations of energy momentum tensors and equations of motion

Here, we consider the generic expression of the perturbations of the energy momentum tensors and equations of motion for a perfect fluid (§3.1), an imperfect fluid (§3.2), and a single scalar field (§3.3). We derive these perturbative expressions in terms of gauge-invariant variables defined in the last section. We also show that all perturbative variables are given in the same form as Eqs. (2.20) and (2.21).

We note that we do not explicitly use any background values of the metric and matter fields in the derivations within this section. Further, we also note that we do not use any information of the Einstein equation nor the equations of state of the matter fields. Therefore, the ingredients of this section will be valid for any background spacetime and many perturbation theories of gravity with general covariance if the decomposition formula (2.8) is correct.

3.1. Perfect fluid

The perturbative expressions of the energy momentum tensor for a perfect fluid are already discussed in KN2007.⁵⁾ Therefore, in this subsection, we just summarize

the definitions and results in KN2007 for the perturbations of the energy momentum tensor in §3.1.1. In addition to the results in KN2007, we also show the perturbative expression of the equations of motion, i.e., the equation of continuity (§3.1.2) and the Euler equation (§3.1.3), which are derived from the perturbations of the divergence of the energy momentum tensor.

3.1.1. Perturbations of the energy momentum tensor

The perturbative expansion of the energy momentum tensor (2.22) is given by

$${}^{(p)}\bar{T}_a{}^b =: {}^{(p)}T_a{}^b + \lambda {}^{(1)}T_a{}^b + \frac{1}{2}\lambda^2 {}^{(2)}T_a{}^b + O(\lambda^3). \quad (3.1)$$

The background energy momentum tensor for a perfect fluid is given by

$${}^{(p)}T_a{}^b = (\epsilon + p)u_a u^b + p\delta_a{}^b. \quad (3.2)$$

The first- and the second-order perturbations ${}^{(1)}T_a{}^b$ and ${}^{(2)}T_a{}^b$ of the energy momentum tensor are also decomposed into the form as Eqs. (2.20) and (2.21), respectively, i.e.,

$${}^{(1)}T_a{}^b =: {}^{(1)}\mathcal{T}_a{}^b + \mathcal{L}_X {}^{(1)}T_a{}^b, \quad (3.3)$$

$${}^{(2)}T_a{}^b =: {}^{(2)}\mathcal{T}_a{}^b + 2\mathcal{L}_X {}^{(1)}T_a{}^b + \{\mathcal{L}_Y - \mathcal{L}_X^2\} {}^{(1)}T_a{}^b, \quad (3.4)$$

where gauge-invariant parts ${}^{(1)}\mathcal{T}_a{}^b$ and ${}^{(2)}\mathcal{T}_a{}^b$ of the first- and the second-order perturbations are given by

$${}^{(1)}\mathcal{T}_a{}^b := \left(\mathcal{E} + \mathcal{P} \right) u_a u^b + \mathcal{P} \delta_a{}^b + (\epsilon + p) \left(u_a \mathcal{U}^b - \mathcal{H}^{bc} u_c u_a + \mathcal{U}_a u^b \right), \quad (3.5)$$

$$\begin{aligned} {}^{(2)}\mathcal{T}_a{}^b := & \left(\mathcal{E}^{(2)} + \mathcal{P}^{(2)} \right) u_a u^b + 2 \left(\mathcal{E}^{(1)} + \mathcal{P}^{(1)} \right) u_a \left(\mathcal{U}^b - \mathcal{H}^{bc} u_c \right) + 2 \left(\mathcal{E}^{(1)} + \mathcal{P}^{(1)} \right) \mathcal{U}_a u^b \\ & + (\epsilon + p) u_a \left(\mathcal{U}^b - 2\mathcal{H}^{bc} \mathcal{U}_c + 2\mathcal{H}^{bc} \mathcal{H}_{cd} u^d - \mathcal{L}^{bd} u_d \right) \\ & + 2(\epsilon + p) \mathcal{U}_a \left(\mathcal{U}^b - \mathcal{H}^{bc} u_c \right) + (\epsilon + p) \mathcal{U}_a u^b + \mathcal{P}^{(2)} \delta_a{}^b, \end{aligned} \quad (3.6)$$

where we defined

$$\mathcal{U}^a := g^{ab} \mathcal{U}_b, \quad \mathcal{U}^a := g^{ab} \mathcal{U}_b. \quad (3.7)$$

We also note that the fluid four-velocities \bar{u}_a and u_a should satisfy the normalization conditions of the four-velocity

$$\bar{g}^{ab} \bar{u}_a \bar{u}_b = g^{ab} u_a u_b = -1. \quad (3.8)$$

These normalization conditions yield

$$u^a \mathcal{U}_a^{(1)} = \frac{1}{2} \mathcal{H}_{ab} u^a u^b, \quad (3.9)$$

$$u^a \mathcal{U}_a^{(2)} = -g_{cb} \left(\mathcal{U}^{(1)b} - \mathcal{H}^{db} u_d \right) \left(\mathcal{U}^{(1)c} - \mathcal{H}^{ac} u_a \right) + \frac{1}{2} \mathcal{L}_{ab} u^a u^b. \quad (3.10)$$

We have to emphasize that the perturbative expressions (3.3) and (3.4) are not definitions but the results which are derived from the definitions (2.26)–(2.28). These are natural results from general formulae (2.20) and (2.21). However, these results imply that the framework developed in KN2003 and KN2005 does work in the case of the perturbations of the energy momentum tensor of a perfect fluid.

3.1.2. Perturbations of the continuity equation

Here, we consider the perturbations of the continuity equation for the perfect fluid which is derived from $\bar{u}^a \bar{\nabla}_b^{(p)} \bar{T}_a^b = 0$. This equation yields

$$\bar{C}_0^{(p)} := \bar{u}^a \bar{\nabla}_a \bar{\epsilon} + (\bar{\epsilon} + \bar{p}) \bar{\theta} = 0. \quad (3.11)$$

The energy density $\bar{\epsilon}$ and the pressure \bar{p} are expanded as Eqs. (2.23), (2.24), respectively. Further, as shown in Appendix A, the four-velocity \bar{u}^a and the expansion $\bar{\theta}$ associated with the four-velocity \bar{u}_a are expanded as Eq. (A.46) and (A.85). Through these equations, we obtain the perturbative expansion of the continuity equation as

$$\bar{C}_0^{(p)} = C_0^{(p)} + \lambda^{(1)} C_0^{(p)} + \frac{1}{2} \lambda^{2(2)} C_0^{(p)} + O(\lambda^3) = 0, \quad (3.12)$$

where the continuity equation of each order is given as follows:

$$C_0^{(p)} := u^a \nabla_a \epsilon + (\epsilon + p) \theta = 0, \quad (3.13)$$

$$^{(1)}C_0^{(p)} := u^a \nabla_a \epsilon^{(1)} + (u^a) \nabla_a \epsilon + \left(\epsilon^{(1)} + p^{(1)} \right) \theta + \frac{(1)}{\theta} (\epsilon + p) = 0, \quad (3.14)$$

$$\begin{aligned} ^{(2)}C_0^{(p)} := & u^a \nabla_a \epsilon^{(2)} + 2 (u^a) \nabla_a \epsilon^{(1)} + (u^a) \nabla_a \epsilon \\ & + \theta \left(\epsilon^{(2)} + p^{(2)} \right) + 2 \frac{(1)}{\theta} \left(\epsilon^{(1)} + p^{(1)} \right) + \frac{(2)}{\theta} (\epsilon + p) = 0. \end{aligned} \quad (3.15)$$

The gauge-invariant variables for each order perturbations of $\bar{\epsilon}$, \bar{p} , \bar{u}^a , and $\bar{\theta}$ are also given by Eqs. (2.26)–(2.28), (A.48), (A.49), (A.89)–(A.92). In terms of these gauge-invariant variables, the first- and the second-order perturbations (3.14) and (3.15) of the continuity equation are given in the gauge-invariant form. First, we derive the gauge-invariant expression of the first-order perturbation (3.14) of the continuity equation (3.11). Substituting Eqs. (2.26), (A.48), and (A.89) into Eq. (3.14), we can decompose the first-order perturbation $^{(1)}C_0^{(p)}$ into the gauge-invariant and gauge-variant parts as

$$^{(1)}C_0^{(p)} = ^{(1)}\mathcal{C}_0^{(p)} + \mathcal{L}_X C_0^{(p)}, \quad (3.16)$$

where

$${}^{(1)}\mathcal{C}_0^{(p)} := u^a \nabla_a {}^{(1)}\mathcal{E} + \left(\mathcal{U}^a - \mathcal{H}^{ab} u_b \right) \nabla_a \epsilon + \left({}^{(1)}\mathcal{E} + {}^{(1)}\mathcal{P} \right) \theta + (\epsilon + p) {}^{(1)}\Theta. \quad (3.17)$$

We note that Eq. (3.16) has the same form as Eq. (2.20). By virtue of the background continuity equation (3.13), the first-order perturbation (3.14) of the continuity equation is given in a gauge-invariant form:

$${}^{(1)}\mathcal{C}_0^{(p)} = 0. \quad (3.18)$$

Further, through the definitions (2.26)–(2.28) and the decomposition formulae (A.48), (A.49), (A.89), and (A.91), the second-order perturbation ${}^{(2)}C_0^{(p)}$ of the continuity equation defined by (3.15) is decomposed into the form

$${}^{(2)}C_0^{(p)} = {}^{(2)}\mathcal{C}_0^{(p)} + 2\mathcal{L}_X {}^{(1)}C_0^{(p)} + (\mathcal{L}_Y - \mathcal{L}_X^2) C_0^{(p)}, \quad (3.19)$$

where

$$\begin{aligned} {}^{(2)}\mathcal{C}_0^{(p)} = & u^a \nabla_a {}^{(2)}\mathcal{E} + \left(\mathcal{U}^a - 2\mathcal{H}^{ab} \mathcal{U}_b + 2\mathcal{H}^{ac} \mathcal{H}_{cb} u^b - \mathcal{L}^{ab} u_b \right) \nabla_a \epsilon \\ & + \theta \left({}^{(2)}\mathcal{E} + {}^{(2)}\mathcal{P} \right) + {}^{(2)}\Theta (\epsilon + p) + 2 \left(\mathcal{U}^a - \mathcal{H}^{ab} u_b \right) \nabla_a {}^{(1)}\mathcal{E} \\ & + 2 {}^{(1)}\Theta \left({}^{(1)}\mathcal{E} + {}^{(1)}\mathcal{P} \right). \end{aligned} \quad (3.20)$$

We also note that Eq. (3.19) has the same form as Eq. (2.21). Through the background equation (3.13) and the first-order perturbation (3.14) of the continuity equation, the second-order perturbation (3.15) of the continuity equation is given in the gauge-invariant form:

$${}^{(2)}\mathcal{C}_0^{(p)} = 0. \quad (3.21)$$

Thus, we have obtained the gauge-invariant form of the first- and the second-order perturbations of the continuity equation for a perfect fluid through the lower order equations without any gauge fixing.

3.1.3. Perturbations of the Euler equation

Here, we consider the perturbations of the Euler equations. The component of $\bar{\nabla}_b {}^{(p)}\bar{T}_a{}^b = 0$ orthogonal to \bar{u}^a gives the Euler equation

$$\bar{C}_b^{(p)} := (\bar{\epsilon} + \bar{p}) \bar{a}_b + \bar{g}^{ac} \bar{q}_{bc} \bar{\nabla}_a \bar{p} = 0. \quad (3.22)$$

Here, the three-metric \bar{q}_{ab} , the acceleration vector \bar{a}_b associated with the four-velocity \bar{u}_a are defined by Eqs. (A.3), (A.45), and (A.65) in Appendix A. Substituting the perturbative expansions (2.23), (2.24), (A.50), (A.4), and (A.47) into the Euler equation (3.22), we obtain the expansion form of Eq. (3.22) as

$$\bar{C}_b^{(p)} =: C_b^{(p)} + \lambda {}^{(1)}C_b^{(p)} + \frac{1}{2} \lambda^2 {}^{(2)}C_b^{(p)} + O(\lambda^3) = 0. \quad (3.23)$$

Then, we obtain the Euler equation of each order:

$$C_b^{(p)} := (\epsilon + p) a_b + g^{ac} q_{bc} \nabla_a p = 0, \quad (3.24)$$

$$\begin{aligned} {}^{(1)}C_b^{(p)} &:= (\epsilon + p) \binom{(1)}{a_b} + \left(\binom{(1)}{\epsilon} + \binom{(1)}{p} \right) a_b + g^{ac} q_{bc} \nabla_a \binom{(1)}{p} \\ &\quad + g^{ac} \binom{(1)}{q_{bc}} \nabla_a p - h^{ac} q_{bc} \nabla_a p = 0, \end{aligned} \quad (3.25)$$

$$\begin{aligned} {}^{(2)}C_b^{(p)} &:= (\epsilon + p) \binom{(2)}{a_b} + 2 \left(\binom{(1)}{\epsilon} + \binom{(1)}{p} \right) \binom{(1)}{a_b} + \left(\binom{(2)}{\epsilon} + \binom{(2)}{p} \right) a_b + g^{ac} q_{bc} \nabla_a \binom{(2)}{p} \\ &\quad + 2g^{ac} \binom{(1)}{q_{bc}} \nabla_a \binom{(1)}{p} + g^{ac} \binom{(2)}{q_{bc}} \nabla_a p - 2h^{ac} q_{bc} \nabla_a \binom{(1)}{p} \\ &\quad - 2h^{ac} \binom{(1)}{q_{bc}} \nabla_a p + \left(2h^{ad} h_d^c - l^{ac} \right) q_{bc} \nabla_a p = 0. \end{aligned} \quad (3.26)$$

As in the case of the continuity equation, the first- and the second-order perturbations (3.25) and (3.26) of the Euler equation should be given in the gauge-invariant form. First, we consider the gauge-invariant expression of the first-order perturbation (3.25) of the Euler equation. Substituting Eqs. (2.8), (2.26), (A.8), and (A.54), we obtain the expression of ${}^{(1)}C_b^{(p)}$ as

$${}^{(1)}C_b^{(p)} = {}^{(1)}\mathcal{C}_b^{(p)} + \mathcal{L}_X C_b^{(p)}, \quad (3.27)$$

where

$$\begin{aligned} {}^{(1)}\mathcal{C}_b^{(p)} &:= (\epsilon + p) \binom{(1)}{\mathcal{A}_b} + \left(\binom{(1)}{\mathcal{E}} + \binom{(1)}{\mathcal{P}} \right) a_b + g^{ac} q_{bc} \nabla_a \binom{(1)}{\mathcal{P}} \\ &\quad + g^{ac} \binom{(1)}{\mathcal{Q}_{bc}} \nabla_a p - g^{ad} g^{ce} \mathcal{H}_{de} q_{bc} \nabla_a p, \end{aligned} \quad (3.28)$$

The equation (3.27) has the same form as Eq. (2.20). By virtue of the background Euler equation (3.24), the first-order perturbation (3.25) of the Euler equation is given in a gauge-invariant form:

$${}^{(1)}\mathcal{C}_b^{(p)} = 0. \quad (3.29)$$

Second, through the definitions (2.26), (2.27) of gauge invariant variables, and the decomposition formulae (2.8), (2.10), (A.8), (A.9), (A.54), and (A.55) of gauge-invariant variables, the second-order perturbation ${}^{(2)}C_b^{(p)}$ defined by (3.26) is decomposed in the form

$${}^{(2)}C_b^{(p)} = {}^{(2)}\mathcal{C}_b^{(p)} + 2\mathcal{L}_X {}^{(1)}C_b^{(p)} + (\mathcal{L}_Y - \mathcal{L}_X^2) C_b^{(p)}, \quad (3.30)$$

where

$$\begin{aligned} {}^{(2)}\mathcal{C}_b^{(p)} &:= (\epsilon + p) \binom{(2)}{\mathcal{A}_b} + \left(\binom{(2)}{\mathcal{E}} + \binom{(2)}{\mathcal{P}} \right) a_b + g^{ac} q_{bc} \nabla_a \binom{(2)}{\mathcal{P}} + g^{ac} \binom{(2)}{\mathcal{Q}_{bc}} \nabla_a p \\ &\quad - g^{ad} g^{ce} \mathcal{L}_{de} q_{bc} \nabla_a p + 2 \left(\binom{(1)}{\mathcal{E}} + \binom{(1)}{\mathcal{P}} \right) \binom{(1)}{\mathcal{A}_b} + 2g^{ac} \binom{(1)}{\mathcal{Q}_{bc}} \nabla_a \binom{(1)}{\mathcal{P}} \\ &\quad - 2\mathcal{H}^{ac} \binom{(1)}{\mathcal{Q}_{bc}} \nabla_a p + 2\mathcal{H}^{ad} g^{ce} \mathcal{H}_{de} q_{bc} \nabla_a p. \end{aligned} \quad (3.31)$$

The equation (3.30) has the same form as Eq. (2.21). Through the background equation (3.24) and the first-order perturbation (3.25) of the Euler equation, the second-order perturbation (3.26) of the Euler equation is given in the gauge-invariant form:

$$^{(2)}\mathcal{C}_b^{(p)} = 0. \quad (3.32)$$

Thus, we have obtained the gauge-invariant form of the first- and the second-order perturbations of the Euler equation for a perfect fluid without any gauge fixing.

3.2. Imperfect fluid

In this subsection, we consider the perturbative expressions of the energy momentum tensor and the equations of motion for an imperfect fluid which includes important effects in the recent cosmology. We derive the gauge-invariant part of the perturbations of the energy-momentum tensor (2.29) for an imperfect fluid in §3.2.1. Then, we derive the gauge-invariant continuity equation for an imperfect fluid §3.2.2. Further, we derive the generalized Navier-Stokes equation in §3.2.3, which corresponds to the Euler equation for a perfect fluid.

3.2.1. Energy momentum tensor

To derive the perturbations of the energy momentum tensor for an imperfect fluid, it is convenient to consider the perturbation of the contravariant energy flux $\bar{q}^a := \bar{g}^{ab}q_b$:

$$\bar{q}^a =: q^a + \lambda \overset{(1)}{(q^a)} + \frac{1}{2}\lambda^2 \overset{(2)}{(q^a)} + O(\lambda^3). \quad (3.33)$$

The perturbations $\overset{(1)}{(q^a)}$ and $\overset{(2)}{(q^a)}$ are given by the same procedure as the derivations of the perturbations of \bar{u}^a in Appendix A.3 and these are decomposed into gauge-invariant and gauge-variant parts:

$$\overset{(1)}{(q^a)} = \overset{(1)}{\mathcal{Q}}^a - \mathcal{H}^{ab}q_b + \mathcal{L}_X q^a, \quad (3.34)$$

$$\overset{(2)}{(q^a)} = \overset{(2)}{\mathcal{Q}}^a - q_b \mathcal{L}^{ab} - 2\mathcal{H}^{ab} \overset{(1)}{\mathcal{Q}}_b + 2\mathcal{H}^{ac}\mathcal{H}_{cd}q^d + 2\mathcal{L}_X \overset{(1)}{(q^a)} + \mathcal{L}_Y q^a - \mathcal{L}_X^2 q^a, \quad (3.35)$$

where we defined

$$\overset{(1)}{\mathcal{Q}}^a := g^{ab} \overset{(1)}{\mathcal{Q}}_b, \quad \overset{(2)}{\mathcal{Q}}^a := g^{ac} \overset{(2)}{\mathcal{Q}}_c. \quad (3.36)$$

Further, $\bar{\pi}_a{}^b := \bar{g}^{bc}\bar{\pi}_{ac}$ is expanded as

$$\bar{\pi}_a{}^b =: \pi_a{}^b + \lambda \overset{(1)}{(\pi_a{}^b)} + \frac{1}{2}\lambda^2 \overset{(2)}{(\pi_a{}^b)} + O(\lambda^3), \quad (3.37)$$

and the similar procedure to decompose the perturbations of \bar{u}^a into the gauge-invariant and the gauge-variant parts yields

$$\overset{(1)}{(\pi_a{}^b)} = \overset{(1)}{\Pi}_a{}^b - \mathcal{H}^{bc}\pi_{ac} + \mathcal{L}_X \pi_a{}^b, \quad (3.38)$$

$$\begin{aligned}
 (\pi_a^b)^{(2)} &= \Pi_a^{(2)b} - 2\mathcal{H}^{cb} \Pi_{ca}^{(1)} - \pi_{ac} \mathcal{L}^{cb} + 2\mathcal{H}^{cd} \mathcal{H}_d^b \pi_{ac} \\
 &\quad + 2\mathcal{L}_X (\pi_a^b)^{(1)} + \mathcal{L}_Y \pi_a^b - \mathcal{L}_X^2 \pi_a^b,
 \end{aligned} \tag{3.39}$$

where we defined

$$\Pi_a^{(1)b} := g^{bc} \Pi_{ac}^{(1)}, \quad \Pi_a^{(2)b} := g^{cb} \Pi_{ca}^{(2)}. \tag{3.40}$$

The perturbative expansions of the traceless property [the last equation in Eqs.(2.32)] of the anisotropic stress $\bar{\pi}_a^b$ are given in gauge-invariant forms as

$$\pi_a^a = 0, \quad \Pi_a^{(1)a} = \mathcal{H}^{ac} \pi_{ac}, \quad \Pi_a^{(2)a} = \pi_{ac} \mathcal{L}^{ca} + 2\mathcal{H}^{ca} \Pi_{ca}^{(1)} - 2\mathcal{H}^{cd} \mathcal{H}_d^a \pi_{ac}, \tag{3.41}$$

where we used $\pi_a^a = (\pi_a^b)^{(1)} = (\pi_a^b)^{(2)} = 0$.

The orthogonal condition (2.31) of the energy flux \bar{q}_a to the four-velocity \bar{u}_a is also expanded perturbatively through Eqs. (2.34) and (A.46). The perturbations of the orthogonal condition (2.31) are given in the gauge-invariant form as follows:

$$u^a q_a = 0, \tag{3.42}$$

$$u^a \mathcal{Q}_a^{(1)} = -q_a \mathcal{U}^a + q_a \mathcal{H}^{ab} u_b, \tag{3.43}$$

$$\begin{aligned}
 u^a \mathcal{Q}_a^{(2)} &= -q_a \mathcal{U}^a + 2q_a \mathcal{H}^{ab} \mathcal{U}_b^{(1)} - 2q_a \mathcal{H}^{ac} \mathcal{H}_{cb} u^b + q_a \mathcal{L}^{ab} u_b \\
 &\quad - 2 \mathcal{Q}_a^{(1)} \mathcal{U}^a + 2 \mathcal{Q}_a^{(1)} \mathcal{H}^{ab} u_b.
 \end{aligned} \tag{3.44}$$

Here, we have to emphasize that we did not fix any gauge choice. The perturbations of the orthogonal conditions (2.31) are also decomposed into the gauge-invariant and the gauge-variant parts as the formulae (2.20) and (2.21). Since the gauge-variant parts of the perturbations of Eq. (2.31) are given by the Lie derivative of its background value and the first-order perturbation of Eq. (2.31), the perturbations of Eq. (2.31) are necessarily given in the gauge-invariant form through the lower order perturbations of Eq. (2.31).

Similarly, the orthogonal condition [the second equation in Eqs. (2.32)] of the anisotropic stress $\bar{\pi}_{ab}$ to the four-velocity \bar{u}^a are also perturbatively expanded through Eqs. (A.46) and (2.35) and these are given by

$$u^a \pi_{ab} = 0, \tag{3.45}$$

$$u^a \Pi_{ab}^{(1)} = -\mathcal{U}^a \pi_{ab} + \mathcal{H}^{ac} u_c \pi_{ab}, \tag{3.46}$$

$$\begin{aligned}
 u^a \Pi_{ab}^{(2)} &= -\mathcal{U}^a \pi_{ab} + 2\mathcal{H}^{ac} \mathcal{U}_c^{(1)} \pi_{ab} - 2\mathcal{H}^{ac} \mathcal{H}_{cd} u^d \pi_{ab} + \mathcal{L}^{ac} u_c \pi_{ab} \\
 &\quad - 2 \Pi_{ab}^{(1)} \left(\mathcal{U}^a - \mathcal{H}^{ab} u_b \right),
 \end{aligned} \tag{3.47}$$

where we have used the background orthogonal condition (3.45) for the anisotropic stress u_a and its first-order perturbation. Eqs. (3.46) and (3.47) are gauge-invariant. This is due to the fact that the gauge-variant parts of the perturbations of $\bar{u}^a \bar{\pi}_{ab}$ are given by the Lie derivative of the lower order perturbations of $\bar{u}^a \bar{\pi}_{ab}$ as Eqs. (2.20) and (2.21) and these gauge-variant parts vanish as in the case of the perturbations of the orthogonal condition $\bar{u}^a \bar{q}_a = 0$.

These orthogonal conditions (3.42)–(3.47) for the perturbations of the energy flux and the anisotropic stress are necessary to specify the independent components of the gauge-invariant parts of the perturbations of the energy flux and the anisotropic stress.

Now, we consider the first- and the second-order perturbations of the imperfect part ${}^{(i)}\bar{T}_a{}^b$ of the energy momentum tensor for an imperfect fluid. Through Eqs. (2.25), (A.46), (2.34), (3.33), and (3.37), the perturbative expansion of the imperfect part (2.33) of the energy momentum tensor is expanded as

$${}^{(i)}\bar{T}_a{}^b = {}^{(i)}T_a{}^b + \lambda {}^{(i)}T_a{}^b + \frac{1}{2} \lambda^2 {}^{(i)}T_a{}^b + O(\lambda^3), \quad (3.48)$$

where

$${}^{(i)}T_a{}^b = q_a u^b + u_a q^b + \pi_a{}^b, \quad (3.49)$$

$${}^{(1)}T_a{}^b = q_a (u^b) + (q_a) u^b + u_a (q^b) + (u_a) q^b + (\pi_a{}^b), \quad (3.50)$$

$$\begin{aligned} {}^{(2)}T_a{}^b = & q_a (u^b) + 2 (q_a) (u^b) + (q_a) u^b + u_a (q^b) + 2 (u_a) (q^b) + (u_a) q^b \\ & + (\pi_a{}^b). \end{aligned} \quad (3.51)$$

Substituting (2.26), (A.48), (2.36), (3.34), and (3.38), into (3.50), the first-order perturbation of the imperfect part of the energy momentum tensor for an imperfect fluid is decomposed as

$${}^{(i)}T_a{}^b = {}^{(i)}\mathcal{T}_a{}^b + \mathcal{L}_X {}^{(i)}T_a{}^b, \quad (3.52)$$

where

$${}^{(i)}\mathcal{T}_a{}^b = q_a \mathcal{U}^b + \mathcal{U}_a q^b + \mathcal{Q}_a u^b + u_a \mathcal{Q}^b - 2u_{(a} q_{c)} \mathcal{H}^{bc} + \Pi_a{}^b - \pi_{ac} \mathcal{H}^{bc}. \quad (3.53)$$

Further, through Eqs. (2.26)–(2.28), (3.34)–(3.36), (3.39), (A.48), and (A.49), the second-order perturbation (3.51) of the imperfect part of the energy momentum tensor for an imperfect fluid is decomposed as

$${}^{(2)}T_a{}^b = {}^{(2)}\mathcal{T}_a{}^b + 2\mathcal{L}_X {}^{(1)}T_a{}^b + (\mathcal{L}_Y - \mathcal{L}_X^2) {}^{(i)}T_a{}^b, \quad (3.54)$$

where

$${}^{(2)}\mathcal{T}_a{}^b = q_a \mathcal{U}^b - 2q_a \mathcal{H}^{bc} \mathcal{U}_c + 2q_a \mathcal{H}^{bd} \mathcal{H}_{dc} u^c - q_a \mathcal{L}^{bc} u_c + \mathcal{Q}_a u^b$$

$$\begin{aligned}
 & -u_a q_c \mathcal{L}^{bc} - 2u_a \mathcal{H}^{bc} \mathcal{Q}_c + 2u_a \mathcal{H}^{bc} \mathcal{H}_{cd} q^d \\
 & + u_a \mathcal{Q}^b + \mathcal{U}_a q^b + \Pi_a^b - 2\mathcal{H}^{cb} \Pi_{ca} - \pi_{ac} \mathcal{L}^{cb} + 2\mathcal{H}^{cd} \mathcal{H}_d^b \pi_{ac} \\
 & + 2 \mathcal{Q}_a \mathcal{U}^b - 2\mathcal{H}^{bc} u_c \mathcal{Q}_a + 2 \mathcal{Q}^b \mathcal{U}_a - 2\mathcal{H}^{bc} q_c \mathcal{U}_a.
 \end{aligned} \tag{3.55}$$

Note that Eqs. (3.52) and (3.54) have the same form as Eqs. (2.20) and (2.21), respectively.

Together with Eqs. (3.3) and (3.4) for the perfect fluid, the total energy momentum tensor of each order is defined by

$$\bar{T}_a^b := T_a^b + \lambda T_a^{(1)b} + \frac{1}{2} \lambda^2 T_a^{(2)b} + O(\lambda^3), \tag{3.56}$$

$$T_a^b := {}^{(p)}T_a^b + {}^{(i)}T_a^b, \quad T_a^{(1)b} := {}^{(p)}T_a^{(1)b} + {}^{(i)}T_a^{(1)b}, \quad T_a^{(2)b} := {}^{(p)}T_a^{(2)b} + {}^{(i)}T_a^{(2)b}. \tag{3.57}$$

Further, the first- and the second-order perturbations of the total energy momentum tensor are also decomposed into the gauge-variant and gauge-invariant parts:

$$T_a^{(1)b} = \mathcal{T}_a^{(1)b} + \mathcal{L}_X T_a^{(1)b}, \tag{3.58}$$

$$T_a^{(2)b} = \mathcal{T}_a^{(2)b} + 2\mathcal{L}_X T_a^{(1)b} + (\mathcal{L}_Y - \mathcal{L}_X^2) T_a^{(1)b}. \tag{3.59}$$

Through Eqs. (3.5) and (3.53), the gauge-invariant part $\mathcal{T}_a^{(1)b}$ of the first-order perturbation of the total energy momentum tensor is given by

$$\mathcal{T}_a^{(1)b} := {}^{(p)}\mathcal{T}_a^{(1)b} + {}^{(i)}\mathcal{T}_a^{(1)b} \tag{3.60}$$

$$\begin{aligned}
 & = \left(\mathcal{E} + \mathcal{P} \right) u_a u^b + \mathcal{P} \delta_a^b + (\epsilon + p) \left(u_a \mathcal{U}^b - \mathcal{H}^{bc} u_c u_a + \mathcal{U}_a u^b \right) \\
 & + q_a \mathcal{U}^b + \mathcal{U}_a q^b + \mathcal{Q}_a u^b + u_a \mathcal{Q}^b - 2u_{(a} q_{c)} \mathcal{H}^{bc} \\
 & + \Pi_a^b - \pi_{ac} \mathcal{H}^{bc}.
 \end{aligned} \tag{3.61}$$

On the other hand, through Eqs. (3.6), and (3.55), the gauge-invariant part $\mathcal{T}_a^{(2)b}$ of the second-order perturbation of the total energy momentum tensor is given by

$$\begin{aligned}
 \mathcal{T}_a^{(2)b} & := {}^{(p)}\mathcal{T}_a^{(2)b} + {}^{(i)}\mathcal{T}_a^{(2)b} \\
 & = \left(\mathcal{E} + \mathcal{P} \right) u_a u^b + 2 \left(\mathcal{E} + \mathcal{P} \right) u_a \left(\mathcal{U}^b - \mathcal{H}^{bc} u_c \right) + 2 \left(\mathcal{E} + \mathcal{P} \right) \mathcal{U}_a u^b
 \end{aligned} \tag{3.62}$$

$$\begin{aligned}
& + (\epsilon + p) u_a \left(\mathcal{U}^b - 2\mathcal{H}^{bc} \mathcal{U}_c^{(1)} + 2\mathcal{H}^{bc} \mathcal{H}_{cd} u^d - \mathcal{L}^{bd} u_d \right) \\
& + 2(\epsilon + p) \mathcal{U}_a^{(1)} \left(\mathcal{U}^b - \mathcal{H}^{bc} u_c \right) + (\epsilon + p) \mathcal{U}_a^{(2)} u^b + \mathcal{P}^{(2)} \delta_a^b. \\
& + q_a \mathcal{U}^b - 2q_a \mathcal{H}^{bc} \mathcal{U}_c^{(1)} + 2q_a \mathcal{H}^{bd} \mathcal{H}_{dc} u^c - q_a \mathcal{L}^{bc} u_c + \mathcal{Q}_a^{(2)} u^b \\
& - u_a q_c \mathcal{L}^{bc} - 2u_a \mathcal{H}^{bc} \mathcal{Q}_c^{(1)} + 2u_a \mathcal{H}^{bc} \mathcal{H}_{cd} q^d + u_a \mathcal{Q}^b + \mathcal{U}_a^{(2)} q^b \\
& + \Pi_a^{(2)} - 2\mathcal{H}^{cb} \Pi_{ca}^{(1)} - \pi_{ac} \mathcal{L}^{cb} + 2\mathcal{H}^{cd} \mathcal{H}_d^b \pi_{ac} \\
& + 2 \mathcal{Q}_a^{(1)} \mathcal{U}^b - 2\mathcal{H}^{bc} u_c \mathcal{Q}_a^{(1)} + 2 \mathcal{Q}^b \mathcal{U}_a^{(1)} - 2\mathcal{H}^{bc} q_c \mathcal{U}_a^{(1)}. \tag{3.63}
\end{aligned}$$

3.2.2. Perturbations of the continuity equation

The equation of motion $\bar{\nabla}_b \bar{T}_a^b = 0$ is decomposed into the tangential and normal components to the four-velocity \bar{u}_a . The tangential component of $\bar{\nabla}_b \bar{T}_a^b = 0$ to \bar{u}_a is given by

$$\bar{u}^b \bar{\nabla}_b \bar{\epsilon} + (\bar{\epsilon} + \bar{p}) \bar{\theta} + \bar{q}^b \bar{a}_b + \bar{\nabla}_b \bar{q}^b + \bar{\pi}^{ab} \bar{B}_{ab} = 0, \tag{3.64}$$

where we have used Eqs. (A.1), (A.2), and (A.65). Eq. (3.64) is the continuity equation for an imperfect fluid. The normal components of $\bar{\nabla}_b \bar{T}_a^b = 0$ to the fluid four-velocity \bar{u}^a is discussed in §3.2.3.

The first two terms in Eq. (3.64) coincide with $\bar{C}_0^{(p)}$ defined by Eq. (3.11) and the perturbative expression of these two terms are given in §3.1.2. Then, we consider the perturbative expression of the remaining terms in Eq. (3.64):

$$\bar{C}_0^{(i)} := \bar{g}^{ab} (\bar{q}_a \bar{a}_b + \bar{\nabla}_a \bar{q}_b) + \bar{g}^{ac} \bar{g}^{bd} \bar{\pi}_{ab} \bar{B}_{cd}. \tag{3.65}$$

We expand this $\bar{C}_0^{(i)}$ as

$$\bar{C}_0^{(i)} = C_0^{(i)} + \lambda C_0^{(i)(1)} + \frac{1}{2} \lambda^2 C_0^{(i)(2)} + O(\lambda^3). \tag{3.66}$$

The first term $\bar{g}^{ab} \bar{q}_a \bar{a}_b$ in Eq. (3.65) is expanded through Eqs. (2.34), (A.47), and (A.50). The last term $\bar{g}^{ac} \bar{g}^{bd} \bar{\pi}_{ab} \bar{B}_{cd}$ in Eq. (3.65) is expanded through Eqs. (2.35), (A.47), and (A.68). Therefore, to derive the explicit expression of $C_0^{(i)}$, $C_0^{(i)(1)}$, and $C_0^{(i)(2)}$, we have to derive the perturbative expression of the term $\bar{\nabla}_a \bar{q}_b$. As in the case of the perturbations of the tensor \bar{A}_{ab} which is derived in Appendix A.2, we can derive the perturbative expansion of the tensor $\bar{\nabla}_a \bar{q}_b$ as follows:

$$\bar{\nabla}_a \bar{q}_b = \nabla_a q_b + \lambda \left(\nabla_a (q_b)^{(1)} - H_{ab}^c [h] q_c \right)$$

$$\begin{aligned}
 & + \frac{1}{2} \lambda^2 \left(\nabla_a^{(2)} (q_b^{(2)}) - 2H_{ab}{}^c [h] (q_c^{(1)}) - H_{ab}{}^c [l] q_c + 2h^{cd} H_{abd} [h] q_c \right) \\
 & + O(\lambda^3).
 \end{aligned} \tag{3.67}$$

Substituting the expansion formulae (2.34), (2.35), (3.67), (A.47), (A.50), and (A.68), the perturbations $C_0^{(i)}$, $C_0^{(1)}$, and $C_0^{(2)}$ of $\bar{C}_0^{(i)}$ defined by Eq. (3.65) are given by

$$C_0^{(i)} = q^b a_b + \nabla_b q^b + \pi^{ab} B_{ab}, \tag{3.68}$$

$$\begin{aligned}
 C_0^{(1)} = & q^b (a_b^{(1)}) + (q_b^{(1)}) a^b + \nabla^b (q_b^{(1)}) - g^{ab} H_{ba}{}^c [h] q_c - h_{ab} q^a a^b - h_{ab} \nabla^a q^b \\
 & + \pi^{ab} B_{ab}^{(1)} + (\pi_{ab})^{(1)} B^{ab} - 2h_{bd} \pi_a{}^b B^{ad},
 \end{aligned} \tag{3.69}$$

$$\begin{aligned}
 C_0^{(2)} = & q^b (a_b^{(2)}) + (q_b^{(2)}) a^b + \nabla^b (q_b^{(2)}) - q^a a^b l_{ab} - \nabla^a q^b l_{ab} \\
 & + (\pi_{ab})^{(2)} B^{ab} + \pi^{cd} B_{cd}^{(2)} - 2l_{bd} \pi^{ab} B_a{}^d - g^{ab} H_{ab}{}^c [l] q_c \\
 & + 2g^{ab} (q_a^{(1)})(a_b^{(1)}) - 2g^{ab} H_{ab}{}^c [h] (q_c^{(1)}) + 2g^{ab} h^{cd} H_{abd} [h] q_c - 2h^{ab} q_a (a_b^{(1)}) \\
 & - 2h^{ab} (q_a^{(1)}) a_b - 2h^{ab} \nabla_a (q_b^{(1)}) + 2h^{ab} H_{ab}{}^c [h] q_c + 2h^{ac} h_c{}^b q_a a_b \\
 & + 2h^{ac} h_c{}^b \nabla_a q_b + 2g^{ac} g^{bd} (\pi_{ab})^{(1)} B_{cd}^{(1)} - 4h_b{}^d \pi^{cb} B_{cd}^{(1)} - 4h_d{}^b (\pi_{ab})^{(1)} B^{ad} \\
 & + 4h_{bf} h^{fd} \pi^{cb} B_{cd} + 2h_{ac} h_{bd} \pi^{ab} B^{cd}.
 \end{aligned} \tag{3.70}$$

Through Eqs. (2.8), (2.36), (2.37), (A.29), (A.54), and (A.72), the first-order perturbation $C_0^{(i)}$ of $\bar{C}_0^{(i)}$ is decomposed as

$$C_0^{(i)} = \mathcal{C}_0^{(i)} + \mathcal{L}_X C_0^{(i)}, \tag{3.71}$$

where

$$\begin{aligned}
 \mathcal{C}_0^{(i)} := & q^b \mathcal{A}_b^{(1)} + \mathcal{Q}_b^{(1)} a^b + \nabla^b \mathcal{Q}_b^{(1)} - g^{ab} H_{ab}{}^c [\mathcal{H}] q_c - \mathcal{H}_{ab} q^a a^b - \mathcal{H}_{ab} \nabla^a q^b \\
 & + \pi^{ab} \mathcal{B}_{ab}^{(1)} + \Pi_{ab}^{(1)} B^{ab} - 2\mathcal{H}_{bd} \pi_a{}^b B^{ad}.
 \end{aligned} \tag{3.72}$$

Similarly, through Eqs. (2.8), (2.10), (2.36), (2.37), (A.29), (A.30), (A.54), (A.55), (A.72), and (A.74), the second-order perturbation $C_0^{(2)}$ is decomposed as

$$C_0^{(i)} = \mathcal{C}_0^{(2)} + 2\mathcal{L}_X \mathcal{C}_0^{(1)} + (\mathcal{L}_Y - \mathcal{L}_X^2) C_0^{(i)}, \tag{3.73}$$

where

$$\begin{aligned}
C_0^{(i)} := & q^b \overset{(2)}{\mathcal{A}}_b + \overset{(2)}{\mathcal{Q}}_b a^b + \nabla^b \overset{(2)}{\mathcal{Q}}_b - q^a a^b \mathcal{L}_{ab} - \nabla^a q^b \mathcal{L}_{ab} \\
& + \overset{(2)}{\Pi}_{ab} B^{ab} + \pi^{cd} \overset{(2)}{\mathcal{B}}_{cd} - 2\mathcal{L}_{bd} \pi^{ab} B_a{}^d - g^{ab} H_{ab}{}^c [\mathcal{L}] q_c \\
& + 2g^{ab} \mathcal{H}^{cd} H_{abd} [\mathcal{H}] q_c + 2g^{ab} \overset{(1)}{\mathcal{Q}}_a \overset{(1)}{\mathcal{A}}_b - 2\mathcal{H}^{ab} \overset{(1)}{\mathcal{Q}}_a a_b - 2\mathcal{H}^{ab} q_a \overset{(1)}{\mathcal{A}}_b \\
& - 2\mathcal{H}^{ab} \nabla_a \overset{(1)}{\mathcal{Q}}_b - 2g^{ab} H_{ab}{}^c [\mathcal{H}] \overset{(1)}{\mathcal{Q}}_c + 2\mathcal{H}^{ab} H_{ab}{}^c [\mathcal{H}] q_c + 2\mathcal{H}^{ac} \mathcal{H}_{cb} q_a a^b \\
& + 2\mathcal{H}^{ac} \mathcal{H}_{ab} \nabla_c q^b + 2g^{ac} g^{bd} \overset{(1)}{\Pi}_{ab} \overset{(1)}{\mathcal{B}}_{cd} - 4\mathcal{H}_{be} g^{de} \pi^{cb} \overset{(1)}{\mathcal{B}}_{cd} - 4g^{ac} \mathcal{H}^{bd} \overset{(1)}{\Pi}_{ab} B_{cd} \\
& + 4\mathcal{H}_{bd} \mathcal{H}^{df} \pi^{cb} B_{cf} + 2\mathcal{H}^{bd} \mathcal{H}^{ac} \pi_{ab} B_{cd}. \tag{3.74}
\end{aligned}$$

Here again, we have confirmed that the perturbations $C_0^{(i)}$ and $C_0^{(p)}$ defined by Eqs. (3.69) and (3.70) are decomposed into gauge-invariant and gauge-variant parts as Eqs. (3.71) and (3.73), and these have the same forms as Eqs. (2.20) and (2.21), respectively.

Hence, through Eqs. (3.12), (3.16), (3.19), (3.66), each order perturbation of continuity equation for an imperfect fluid is given by the gauge-invariant form as

$$C_0^{(p)} + C_0^{(i)} = 0, \tag{3.75}$$

$$\begin{aligned}
\overset{(1)}{C}_0^{(p)} + \overset{(1)}{C}_0^{(i)} &= \overset{(1)}{\mathcal{C}}_0^{(p)} + \overset{(1)}{\mathcal{C}}_0^{(i)} + \mathcal{L}_X \left(C_0^{(p)} + C_0^{(i)} \right) \\
&= \overset{(1)}{\mathcal{C}}_0^{(p)} + \overset{(1)}{\mathcal{C}}_0^{(i)} = 0, \tag{3.76}
\end{aligned}$$

$$\begin{aligned}
\overset{(2)}{C}_0^{(p)} + \overset{(2)}{C}_0^{(i)} &= \overset{(2)}{\mathcal{C}}_0^{(p)} + \overset{(2)}{\mathcal{C}}_0^{(i)} + 2\mathcal{L}_X \left(\overset{(1)}{\mathcal{C}}_0^{(p)} + \overset{(1)}{\mathcal{C}}_0^{(i)} \right) + (\mathcal{L}_Y - \mathcal{L}_X^2) \left(C_0^{(p)} + C_0^{(i)} \right) \\
&= \overset{(2)}{\mathcal{C}}_0^{(p)} + \overset{(2)}{\mathcal{C}}_0^{(i)} = 0, \tag{3.77}
\end{aligned}$$

where we used the background continuity equation (3.75) in the derivation of Eq. (3.76), and used the background continuity equation (3.75) and its first-order perturbation (3.76) in the derivation of Eq. (3.77).

3.2.3. Perturbations of the generalized Navier-Stokes equation

The normal component of $\bar{\nabla}_b \bar{T}_a{}^b$ to \bar{u}_a for an imperfect fluid (2.29) is given by

$$(\bar{\epsilon} + \bar{p}) \bar{a}_b + \bar{q}_c{}^a \left(\bar{\nabla}_a \bar{p} + \bar{u}^b \bar{\nabla}_b \bar{q}_a + \bar{\nabla}_b \bar{\pi}_a{}^b \right) + \bar{q}^b (\bar{q}_{cb} \bar{\theta} + \bar{B}_{cb}) = 0. \tag{3.78}$$

This equation (3.78) is an extension of the Euler equation (3.22) for a perfect fluid to the imperfect fluid case. In this paper, we call Eq. (3.78) as “*the generalized Navier-Stokes equation*”.¹⁸⁾

As seen in the case of the Euler equation (3.22) for a perfect fluid, the first two terms of the generalized Navier-Stokes equation (3.78) is given by $\bar{C}_c^{(p)}$ which

is defined in Eq. (3.22). We call this $\bar{C}_c^{(p)}$ as “the perfect part” of the generalized Navier-Stokes equation (3.78). Using this perfect part, the equation (3.78) is given by

$$\bar{C}_c^{(p)} + \bar{C}_c^{(i)} = 0, \quad (3.79)$$

where $\bar{C}_c^{(i)}$ is defined by

$$\bar{C}_c^{(i)} := \bar{g}^{ad} \bar{q}_{cd} \left(\bar{u}^b \bar{\nabla}_b \bar{q}_a + \bar{g}^{be} \bar{\nabla}_b \bar{\pi}_{ae} \right) + \bar{g}^{be} \bar{q}_e \left(\bar{q}_{bc} \bar{\theta} + \bar{B}_{bc} \right). \quad (3.80)$$

We call this $\bar{C}_c^{(i)}$ as “the imperfect part” of the generalized Navier-Stokes equation for an imperfect fluid. Since the perturbative expansion of the perfect part $\bar{C}_c^{(p)}$ is already given in §3.1.3, we may concentrate on the perturbative expansion of the imperfect part (3.80).

As in the cases of the imperfect part of the continuity equation for an imperfect fluid, the imperfect part of the Navier-Stokes equation is perturbatively expanded as

$$\bar{C}_b^{(i)} =: C_b^{(i)} + \lambda C_b^{(i)(1)} + \frac{1}{2} \lambda^2 C_b^{(i)(2)} + O(\lambda^3). \quad (3.81)$$

Here, $C_b^{(i)(1)}$, $C_b^{(i)(2)}$, and $C_b^{(i)}$ are given by the perturbative expansion of Eq. (3.80) through Eqs. (2.34), (3.67), (A.4), (A.46), (A.47), (A.68), (A.85), and the perturbation of the tensor $\bar{\nabla}_b \bar{\pi}_{ae}$. To derive the perturbative expression of $\bar{\nabla}_b \bar{\pi}_{ae}$, the perturbative expansion (2.35) of the anisotropic stress and the perturbation (A.25) of the connection C_{ba}^c are used. Further, through Eqs. (2.8), (2.10), (2.36), (2.37), (A.8), (A.9), (A.29), (A.30), (A.72), (A.74), (A.89), (A.91), we can show that the first- and the second-order perturbations of the imperfect part $\bar{C}_b^{(i)}$ are also decomposed into the gauge-invariant and the gauge-variant parts as

$$C_b^{(i)(1)} = \mathcal{C}_b^{(i)(1)} + \mathcal{L}_X C_b^{(i)(1)}, \quad (3.82)$$

$$C_b^{(i)(2)} = \mathcal{C}_b^{(i)(2)} + \mathcal{L}_Y C_b^{(i)(2)} - \mathcal{L}_X^2 C_b^{(i)(2)} + 2\mathcal{L}_X C_b^{(i)(1)}. \quad (3.83)$$

These have the same form as Eqs. (2.20) and (2.21), respectively.

The derivations of the gauge-invariant part of $\mathcal{C}_b^{(i)(1)}$ and $\mathcal{C}_b^{(i)(2)}$ are straightforward. Therefore, we just summarize the results, here. The details of these derivations can be seen in the Appendix in Ref. 19). Through Eqs. (3.24)–(3.28), (3.30), and (3.31), each order perturbation of the generalized Navier-Stokes equation for an imperfect fluid is summarized as follows. The background generalized Navier-Stokes equation is given by

$$C_b^{(p)} + C_b^{(i)} = (\epsilon + p) a_b + g^{ac} q_{bc} \nabla_a p + q_{bd} \left(u^c \nabla_c q^d + \nabla_c \pi^{dc} \right) + q^c (q_{cb} \theta + B_{cb}) \quad (3.84)$$

$$= 0. \quad (3.85)$$

The first-order perturbation of the generalized Navier-Stokes equation is given by

$$C_b^{(p)} + C_b^{(i)} = C_b^{(p)} + C_b^{(i)} + \mathcal{L}_X \left(C_b^{(p)} + C_b^{(i)} \right), \quad (3.86)$$

where

$$\begin{aligned} C_b^{(p)} + C_b^{(i)} = & (\epsilon + p) \mathcal{A}_b + \left(\mathcal{E} + \mathcal{P} \right) a_b + g^{ac} q_{bc} \nabla_a \mathcal{P} \\ & + g^{ac} \mathcal{Q}_{bc} \nabla_a p - g^{ad} g^{ce} \mathcal{H}_{de} q_{bc} \nabla_a p + q_{bd} u^c \nabla_c \mathcal{Q}^d - q_{bd} u_c H^{dcf} [\mathcal{H}] q_f \\ & + q_{bd} \mathcal{U}^c \nabla_c q^d - q_{bd} \mathcal{H}^{ce} u_e \nabla_c q^d + \mathcal{Q}_{bd} u^c \nabla_c q^d - \mathcal{H}^{ad} q_{bd} u^c \nabla_c q_a \\ & - \mathcal{H}^{ad} q_{bd} \nabla^c \pi_{ac} + \mathcal{Q}_{bd} \nabla_c \pi^{dc} - q_{bd} \mathcal{H}_{ce} \nabla^c \pi^{de} \\ & + q_{bd} \nabla_c \Pi^{dc} - q_{bd} H^{dcf} [\mathcal{H}] \pi_{fc} - g^{ad} q_{bd} H_c{}^{cf} [\mathcal{H}] \pi_{af} \\ & - \mathcal{H}^{ce} q_e q_{bc} \theta + \mathcal{Q}^c q_{bc} \theta + q^c \mathcal{Q}_{bc} \theta + q^c q_{bc} \Theta \\ & + \mathcal{Q}^c B_{cb} - \mathcal{H}^{ce} q_e B_{cb} + q^c \mathcal{B}_{cb}. \end{aligned} \quad (3.87)$$

Then, the first-order perturbation of the generalized Navier-Stokes equation is given in the gauge-invariant form as

$$C_b^{(p)} + C_b^{(i)} = 0, \quad (3.88)$$

where we used the background equation (3.85) of the generalized Navier-Stokes equation. The second-order perturbation of the generalized Navier-Stokes equation is given by

$$\begin{aligned} C_b^{(p)} + C_b^{(i)} = & C_b^{(p)} + C_b^{(i)} \\ & + 2\mathcal{L}_X \left(C_b^{(p)} + C_b^{(i)} \right) + \{ \mathcal{L}_Y - \mathcal{L}_X^2 \} \left(C_b^{(p)} + C_b^{(i)} \right), \end{aligned} \quad (3.89)$$

where

$$\begin{aligned} C_b^{(p)} + C_b^{(i)} = & (\epsilon + p) \mathcal{A}_b + \left(\mathcal{E} + \mathcal{P} \right) a_b + g^{ac} q_{bc} \nabla_a \mathcal{P} + g^{ac} \mathcal{Q}_{bc} \nabla_a p \\ & - g^{ad} g^{ce} \mathcal{L}_{de} q_{bc} \nabla_a p + 2 \left(\mathcal{E} + \mathcal{P} \right) \mathcal{A}_b + 2g^{ac} \mathcal{Q}_{bc} \nabla_a \mathcal{P} \\ & - 2\mathcal{H}^{ac} q_{bc} \nabla_a \mathcal{P} - 2\mathcal{H}^{ac} \mathcal{Q}_{bc} \nabla_a p + 2\mathcal{H}^{ad} g^{ce} \mathcal{H}_{de} q_{bc} \nabla_a p \\ & + q_{bd} u^c \nabla_c \mathcal{Q}^d - q_{bd} u_c H^{dcf} [\mathcal{L}] q_f + 2q_{bd} u_c \mathcal{H}_{fg} H^{dcg} [\mathcal{H}] q^f \end{aligned}$$

$$\begin{aligned}
 & +q_{bd} \mathcal{U}^{(2)c} \nabla_c q^d - q_{bd} \mathcal{L}^{ce} u_e \nabla_c q^d + \mathcal{Q}_{bd}^{(2)} u^c \nabla_c q^d - \mathcal{L}^{ad} q_{bd} u^c \nabla_c q_a \\
 & -2q_{bd} u_c H^{dcf} [\mathcal{H}] \mathcal{Q}_f^{(1)} + 2q_{bd} \mathcal{U}^{(1)c} \nabla_c \mathcal{Q}^d - 2q_{bd} \mathcal{U}_c^{(1)} H^{dcf} [\mathcal{H}] q_f \\
 & -2q_{bd} \mathcal{H}^{ce} u_e \nabla_c \mathcal{Q}^d + 2q_{bd} \mathcal{H}_{ce} u^e H^{dcf} [\mathcal{H}] q_f - 2q_{bd} \mathcal{H}^{ce} \mathcal{U}_e^{(1)} \nabla_c q^d \\
 & +2q_{bd} \mathcal{H}_{fg} \mathcal{H}^{cf} u^g \nabla_c q^d + 2 \mathcal{Q}_{bd}^{(1)} u^c \nabla_c \mathcal{Q}^d - 2 \mathcal{Q}_{bd}^{(1)} u_c H^{dcf} [\mathcal{H}] q_f \\
 & +2 \mathcal{Q}_{bd}^{(1)} \mathcal{U}^{(1)c} \nabla_c q^d - 2 \mathcal{Q}_{bd}^{(1)} \mathcal{H}^{ce} u_e \nabla_c q^d - 2\mathcal{H}^{ad} q_{bd} u^c \nabla_c \mathcal{Q}_a^{(1)} \\
 & +2\mathcal{H}^{ad} q_{bd} u^c H_{acf} [\mathcal{H}] q^f - 2\mathcal{H}^{ad} q_{bd} \mathcal{U}^{(1)c} \nabla_c q_a + 2\mathcal{H}^{ad} q_{bd} \mathcal{H}^{ce} u_e \nabla_c q_a \\
 & -2\mathcal{H}^{ad} \mathcal{Q}_{bd}^{(1)} u^c \nabla_c q_a + 2\mathcal{H}_{af} \mathcal{H}^{fd} q_{bd} u^c \nabla_c q^a - \mathcal{L}^{ad} q_{bd} \nabla^c \pi_{ac} \\
 & + \mathcal{Q}_{bd}^{(2)} \nabla_c \pi^{dc} - q_{bd} \mathcal{L}_{ce} \nabla^c \pi^{de} + q_{bd} \nabla_c \Pi^{dc(2)} \\
 & -q_{bd} H^{dcf} [\mathcal{L}] \pi_{fc} + 2g^{ad} q_{bd} \mathcal{H}_{fg} H_{ac}{}^g [\mathcal{H}] \pi^{fc} - g^{ad} q_{bd} H_c{}^{cf} [\mathcal{L}] \pi_{af} \\
 & +2q_{bd} \mathcal{H}_{fg} H_c{}^{cg} [\mathcal{H}] \pi^{df} - 2\mathcal{H}^{ad} \mathcal{Q}_{bd}^{(1)} \nabla^c \pi_{ac} + 2\mathcal{H}^{ad} q_{bd} \mathcal{H}^{ce} \nabla_c \pi_{ae} \\
 & -2\mathcal{H}^{ad} q_{bd} \nabla^c (\Pi_{ac}^{(1)}) + 2\mathcal{H}^{ad} q_{bd} H_a{}^{cf} [\mathcal{H}] \pi_{fc} + 2\mathcal{H}^{ad} q_{bd} H_c{}^{cf} [\mathcal{H}] \pi_{af} \\
 & -2 \mathcal{Q}_{bd}^{(1)} \mathcal{H}_{ce} \nabla^c \pi^{de} + 2 \mathcal{Q}_{bd}^{(1)} \nabla_c \Pi^{dc(1)} - 2 \mathcal{Q}_{bd}^{(1)} H^{dcf} [\mathcal{H}] \pi_{fc} \\
 & -2g^{ad} \mathcal{Q}_{bd}^{(1)} H_c{}^{cf} [\mathcal{H}] \pi_{af} - 2q_{bd} \mathcal{H}_{ce} \nabla^c \Pi^{de(1)} + 2q_{bd} \mathcal{H}_{ap} \mathcal{H}^{pd} \nabla_c \pi^{ac} \\
 & +2q_{bd} \mathcal{H}^{cq} \mathcal{H}_{qp} \nabla_c \pi^{dp} + 2g^{ad} q_{bd} \mathcal{H}^{ce} H_{ac}{}^f [\mathcal{H}] \pi_{fe} + 2q_{bd} \mathcal{H}^{ce} H_{ecf} [\mathcal{H}] \pi^{df} \\
 & -2q_{bd} H^{dcf} [\mathcal{H}] \Pi_{fc}^{(1)} - 2g^{ad} q_{bd} H_c{}^{cf} [\mathcal{H}] \Pi_{af}^{(1)} - \mathcal{L}^{ce} q_e q_{bc} \theta \\
 & + \mathcal{Q}^{(2)c} q_{bc} \theta + q^c \mathcal{Q}_{bc}^{(2)} \theta + q^c q_{bc} \Theta^{(2)} \\
 & -2\mathcal{H}^{ce} \mathcal{Q}_e^{(1)} q_{bc} \theta + 2\mathcal{H}^{cf} \mathcal{H}_{fe} q^e q_{bc} \theta - 2\mathcal{H}^{ce} q_e \mathcal{Q}_{bc}^{(1)} \theta \\
 & -2\mathcal{H}^{ce} q_e q_{bc} \Theta^{(1)} + 2 \mathcal{Q}^{(1)c} \mathcal{Q}_{bc}^{(1)} \theta + 2 \mathcal{Q}^{(1)c} q_{bc} \Theta^{(1)} \\
 & +2q^c \mathcal{Q}_{bc}^{(1)} \Theta^{(1)} + \mathcal{Q}^{(2)c} B_{cb} - \mathcal{L}^{ce} q_e B_{cb} \\
 & +q^c \mathcal{B}_{cb}^{(2)} - 2\mathcal{H}^{ce} \mathcal{Q}_e^{(1)} B_{cb} - 2\mathcal{H}^{ce} q_e \mathcal{B}_{cb}^{(1)} \\
 & +2 \mathcal{Q}^{(1)c} \mathcal{B}_{cb}^{(1)} + 2\mathcal{H}^{cd} \mathcal{H}_{de} q^e B_{cb}.
 \end{aligned} \tag{3.90}$$

Through Eqs. (3.85) and (3.88), the second-order perturbation of the generalized Navier-Stokes equation is given in the gauge-invariant form as

$$\mathcal{C}_b^{(2)(p)} + \mathcal{C}_b^{(2)(i)} = 0. \tag{3.91}$$

3.3. Scalar fluid

Here, we consider the perturbations of the energy momentum tensor (2·38) and the Klein-Gordon equation of the single scalar field $\bar{\varphi}$. Since the perturbative expression of the energy momentum tensor of the single scalar field is already discussed in KN2007, we just summarize the formulae for the energy momentum tensor in §3.3.1. After that, we consider the perturbations of the Klein-Gordon equation in §3.3.2.

3.3.1. Energy momentum tensor

Through the perturbative expansions (2·39) and (A·47) of the scalar field $\bar{\varphi}$ and the metric, the energy momentum tensor (2·38) is also expanded as

$$\bar{T}_a{}^b = T_a{}^b + \lambda^{(1)}(T_a{}^b) + \frac{1}{2}\lambda^{(2)}(T_a{}^b) + O(\lambda^3). \quad (3.92)$$

The background energy momentum tensor $T_a{}^b$ is given by

$$T_a{}^b := \nabla_a \varphi g^{bc} \nabla_c \varphi - \frac{1}{2} \delta_a{}^b (\nabla_c \varphi \nabla^c \varphi + 2V(\varphi)). \quad (3.93)$$

Further, through the decompositions (2·8), (2·10), (2·40), and (2·41) of the perturbations of the metric and the scalar field, the perturbations of the energy momentum tensor $^{(1)}(T_a{}^b)$ and $^{(2)}(T_a{}^b)$ are also decomposed into the gauge-invariant and the gauge-variant parts as

$$^{(1)}(T_a{}^b) =: {}^{(1)}\mathcal{T}_a{}^b + \mathcal{L}_X T_a{}^b, \quad (3.94)$$

$$^{(2)}(T_a{}^b) =: {}^{(2)}\mathcal{T}_a{}^b + 2\mathcal{L}_X {}^{(1)}(T_a{}^b) + (\mathcal{L}_Y - \mathcal{L}_X^2) T_a{}^b, \quad (3.95)$$

where the gauge-invariant parts $^{(1)}\mathcal{T}_a{}^b$ and $^{(2)}\mathcal{T}_a{}^b$ of the first and the second order are given by

$$\begin{aligned} {}^{(1)}\mathcal{T}_a{}^b &:= \nabla_a \varphi \nabla^b \varphi_1 - \nabla_a \varphi \mathcal{H}^{bc} \nabla_c \varphi + \nabla_a \varphi_1 \nabla^b \varphi \\ &\quad - \frac{1}{2} \delta_a{}^b \left(\nabla_c \varphi \nabla^c \varphi_1 - \nabla_c \varphi \mathcal{H}^{dc} \nabla_d \varphi + \nabla_c \varphi_1 \nabla^c \varphi + 2\varphi_1 \frac{\partial V}{\partial \varphi} \right), \quad (3.96) \\ {}^{(2)}\mathcal{T}_a{}^b &:= \nabla_a \varphi \nabla^b \varphi_2 - 2\nabla_a \varphi \mathcal{H}^{bc} \nabla_c \varphi_1 + 2\nabla_a \varphi \mathcal{H}^{bd} \mathcal{H}_{dc} \nabla^c \varphi - \nabla_a \varphi g^{bd} \mathcal{L}_{dc} \nabla^c \varphi \\ &\quad + 2\nabla_a \varphi_1 \nabla^b \varphi_1 - 2\nabla_a \varphi_1 \mathcal{H}^{bc} \nabla_c \varphi + \nabla_a \varphi_2 \nabla^b \varphi \\ &\quad - \frac{1}{2} \delta_a{}^b \left(\nabla_c \varphi \nabla^c \varphi_2 - 2\nabla_c \varphi \mathcal{H}^{dc} \nabla_d \varphi_1 + 2\nabla^c \varphi \mathcal{H}^{de} \mathcal{H}_{ec} \nabla_d \varphi \right. \\ &\quad \left. - \nabla^c \varphi \mathcal{L}_{dc} \nabla^d \varphi + 2\nabla_c \varphi_1 \nabla^c \varphi_1 - 2\nabla_c \varphi_1 \mathcal{H}^{dc} \nabla_d \varphi \right. \\ &\quad \left. + \nabla_c \varphi_2 \nabla^c \varphi + 2\varphi_2 \frac{\partial V}{\partial \varphi} + 2\varphi_1^2 \frac{\partial^2 V}{\partial \varphi^2} \right). \quad (3.97) \end{aligned}$$

We also note that in these derivations, we did not use the homogeneous condition for the background scalar field φ and the decompositions (3·94) and (3·95) coincide with the formulae (2·20) and (2·21), respectively.

3.3.2. Klein-Gordon equation

Here, we consider the perturbation of the Klein-Gordon equation

$$\bar{C}_{(K)} := \bar{\nabla}^a \bar{\nabla}_a \bar{\varphi} - \frac{\partial V}{\partial \bar{\varphi}}(\bar{\varphi}) = 0. \quad (3.98)$$

Through the perturbative expansions (2.39) and (2.7), the Klein-Gordon equation (3.98) is expanded as

$$\bar{C}_{(K)} =: C_{(K)} + \lambda C_{(K)}^{(1)} + \frac{1}{2} \lambda^2 C_{(K)}^{(2)} + O(\lambda^3), \quad (3.99)$$

where

$$C_{(K)} := \nabla_a \nabla^a \varphi - \frac{\partial V}{\partial \varphi}(\varphi) = 0, \quad (3.100)$$

$$C_{(K)}^{(1)} := \nabla^a \nabla_a \hat{\varphi}_1 - g^{ab} H_{ab}{}^c [h] \nabla_c \varphi - h^{ab} \nabla_a \nabla_b \varphi - \hat{\varphi}_1 \frac{\partial^2 V}{\partial \varphi^2}(\varphi) = 0, \quad (3.101)$$

$$\begin{aligned} C_{(K)}^{(2)} := & \nabla^a \nabla_a \hat{\varphi}_2 - 2h^{ab} \nabla_a \nabla_b \hat{\varphi}_1 - 2g^{ab} H_{ab}{}^c [h] \nabla_c \hat{\varphi}_1 + 2h^{ab} H_{ab}{}^c [h] \nabla_c \varphi \\ & - g^{ab} \left(H_{ab}{}^c [l] - 2h^{cd} H_{abd} [h] \right) \nabla_c \varphi + 2h^{ae} h_e{}^b \nabla_a \nabla_b \varphi - l^{ab} \nabla_a \nabla_b \varphi \\ & - \hat{\varphi}_2 \frac{\partial^2 V}{\partial \varphi^2}(\varphi) - (\hat{\varphi}_1)^2 \frac{\partial^3 V}{\partial \varphi^3}(\varphi) = 0. \end{aligned} \quad (3.102)$$

Here, to derive the perturbative expansion of the kinetic terms of the Klein-Gordon equation (3.98), we use the connection (A.25) between the covariant derivatives $\bar{\nabla}_a$ and ∇_a as in the case of the perturbative expansion of the tensor \bar{A}_{ab} in Appendix A.2.

The first- and the second-order perturbations $C_{(K)}^{(1)}$ and $C_{(K)}^{(2)}$ of the Klein-Gordon equation are also decomposed into the gauge-invariant and the gauge-variant parts. Through Eqs. (2.8), (2.10), (2.40), (2.41), (A.29), and (A.30), the first- and the second-order perturbations (3.101) and (3.102) of the Klein-Gordon equation are decomposed as

$$C_{(K)}^{(1)} =: \mathcal{C}_{(K)}^{(1)} + \mathcal{L}_X C_{(K)}, \quad (3.103)$$

$$C_{(K)}^{(2)} =: \mathcal{C}_{(K)}^{(2)} + 2\mathcal{L}_X C_{(K)}^{(1)} + (\mathcal{L}_Y - \mathcal{L}_X^2) C_{(K)}, \quad (3.104)$$

where

$$\mathcal{C}_{(K)}^{(1)} := \nabla^a \nabla_a \varphi_1 - H_a{}^{ac} [\mathcal{H}] \nabla_c \varphi - \mathcal{H}^{ab} \nabla_a \nabla_b \varphi - \varphi_1 \frac{\partial^2 V}{\partial \varphi^2}(\varphi), \quad (3.105)$$

$$\begin{aligned} \mathcal{C}_{(K)}^{(2)} := & \nabla^a \nabla_a \varphi_2 - H_a{}^{ac} [\mathcal{L}] \nabla_c \varphi + 2H_a{}^{ad} [\mathcal{H}] \mathcal{H}_{cd} \nabla^c \varphi - 2H_a{}^{ac} [\mathcal{H}] \nabla_c \varphi_1 \\ & + 2\mathcal{H}^{ab} H_{ab}{}^c [\mathcal{H}] \nabla_c \varphi - \mathcal{L}^{ab} \nabla_a \nabla_b \varphi + 2\mathcal{H}^a{}_d \mathcal{H}^{db} \nabla_a \nabla_b \varphi - 2\mathcal{H}^{ab} \nabla_a \nabla_b \varphi_1 \\ & - \varphi_2 \frac{\partial^2 V}{\partial \varphi^2}(\varphi) - (\varphi_1)^2 \frac{\partial^3 V}{\partial \varphi^3}(\varphi). \end{aligned} \quad (3.106)$$

Here, we note that Eqs. (3.103) and (3.104) have the same form as Eqs. (2.20) and (2.21), respectively.

By virtue of the background Klein-Gordon equation (3.100) and the first-order perturbation (3.101) of the Klein-Gordon equation, the first- and the second-order perturbation of the Klein-Gordon equation are given in terms of gauge-invariant variables:

$$\mathcal{C}_{(K)}^{(1)} = 0, \quad \mathcal{C}_{(K)}^{(2)} = 0. \quad (3.107)$$

§4. Energy momentum tensors and equations of motion in cosmological situations

Here, we derive the explicit components of the energy momentum tensors and the equations of motion for a perfect fluid, an imperfect fluid, and a scalar field. In the derivation of these equations, we do not use any information of the Einstein equations to guarantee the validity of the formulae to wide applications.

4.1. Perfect fluid

4.1.1. Energy momentum tensor

As shown in §3.1.1, the gauge-invariant parts of the first- and the second-order perturbations of the energy momentum tensor for a perfect fluid are given by Eqs. (3.5) and (3.6). From Eqs. (2.13), (A.12), and (A.13), the components of the first-order perturbation of the energy-momentum tensor of the single perfect fluid are given by

$${}^{(p)}\mathcal{T}_\eta^{(1)} = -\mathcal{E}^{(1)}, \quad {}^{(p)}\mathcal{T}_\eta^{(1)i} = -(\epsilon + p) \left(D^i v^{(1)} + \mathcal{V}^i - \nu^i \right), \quad (4.1)$$

$${}^{(p)}\mathcal{T}_i^{(1)\eta} = (\epsilon + p) \left(D_i v^{(1)} + \mathcal{V}_i \right), \quad {}^{(p)}\mathcal{T}_i^{(1)j} = \mathcal{P}^{(1)} \delta_i^j. \quad (4.2)$$

Further, from the components of the first- and the second-order perturbations of the fluid four-velocity which are given by Eqs. (A.12)–(A.14), the components (2.13) and (2.15) of the first- and the second-order metric perturbations, the components of the second-order perturbation of the energy-momentum tensor of the single perfect fluid are given by

$${}^{(p)}\mathcal{T}_\eta^{(2)} = -\mathcal{E}^{(2)} - 2(\epsilon + p) \left(D_i v^{(1)} + \mathcal{V}_i \right) \left(D^i v^{(1)} + \mathcal{V}^i - \nu^i \right), \quad (4.3)$$

$$\begin{aligned} {}^{(p)}\mathcal{T}_i^{(2)\eta} &= 2 \left(\mathcal{E}^{(1)} + \mathcal{P}^{(1)} \right) \left(D_i v^{(1)} + \mathcal{V}_i \right) \\ &\quad + (\epsilon + p) \left(D_i v^{(2)} + \mathcal{V}_i^{(2)} - 2 \Phi^{(1)} D_i v^{(1)} - 2 \Phi^{(1)} \mathcal{V}_i^{(1)} \right), \end{aligned} \quad (4.4)$$

$$\begin{aligned}
 {}^{(p)}\mathcal{T}_\eta{}^i &= -2 \left(\mathcal{E}^{(1)} + \mathcal{P}^{(1)} \right) \left(D^i v^{(1)} + \mathcal{V}^i - \nu^i \right) \\
 &\quad + (\epsilon + p) \left\{ -D^i v^{(2)} - \mathcal{V}^i + \nu^i - 2 \mathcal{P}^{(1)} \left(D^i v^{(1)} + \mathcal{V}^i \right) \right. \\
 &\quad \left. + 2 \left(-2 \mathcal{P}^{(1)} \gamma^{ij} + \chi^{ij} \right) \left(D_j v^{(1)} + \mathcal{V}_j - \nu_j \right) \right\}, \quad (4.5)
 \end{aligned}$$

$${}^{(p)}\mathcal{T}_i{}^j = 2(\epsilon + p) \left(D_i v^{(1)} + \mathcal{V}_i \right) \left(D^j v^{(1)} + \mathcal{V}^j - \nu^j \right) + \mathcal{P}^{(2)} \delta_i{}^j. \quad (4.6)$$

In Eqs. (4.3)–(4.6), the vector- and the tensor-modes of the first-order perturbations are included into our consideration, which are ignored in KN2007.

4.1.2. Perturbative continuity equation of a perfect fluid

Now, we derive the explicit expression of the first- and the second-order perturbations (3.18) and (3.21) of the continuity equation for a perfect fluid in the cosmological situation. Before the derivation of the perturbations, we first show the background continuity equation (3.13) in the cosmological situation, which is given by

$$aC_0^{(p)} = \partial_\eta \epsilon + 3\mathcal{H}(\epsilon + p) = 0. \quad (4.7)$$

Next, we consider the first-order perturbation (3.17) of the continuity equation:

$$a^{(1)}\mathcal{C}_0^{(p)} = \partial_\eta \mathcal{E}^{(1)} + 3\mathcal{H} \left(\mathcal{E}^{(1)} + \mathcal{P}^{(1)} \right) + (\epsilon + p) \left(\Delta v^{(1)} - 3\partial_\eta \mathcal{P}^{(1)} \right) = 0, \quad (4.8)$$

where we have used Eqs. (2.13), (A.12), (A.13), (A.93), (A.94), $\epsilon = \epsilon(\eta)$, and (4.7).

On the other hand, the explicit expression of the second-order perturbation (3.20) in the cosmological situation is given as follows. Through Eqs. (2.13), (2.15), (A.12)–(A.14), (A.93)–(A.95), $\epsilon = \epsilon(\eta)$, (4.7), and (4.8), the second-order perturbation ${}^{(2)}\mathcal{C}_0^{(p)}$ of the continuity equation is given by

$$a^{(2)}\mathcal{C}_0^{(p)} = \partial_\eta \mathcal{E}^{(2)} + 3\mathcal{H} \left(\mathcal{E}^{(2)} + \mathcal{P}^{(2)} \right) + \left(\Delta v^{(2)} - 3\partial_\eta \mathcal{P}^{(2)} \right) (\epsilon + p) - \Xi_0^{(p)} = 0, \quad (4.9)$$

where

$$\begin{aligned}
 \Xi_0^{(p)} &:= 2 \left[6 \mathcal{P}^{(1)} \partial_\eta \mathcal{V}^{(1)} - \left(2 \mathcal{P}^{(1)} + \mathcal{P}^{(1)} \right) \Delta v^{(1)} - \nu^k D_k \mathcal{P}^{(1)} \right. \\
 &\quad \left. + \left(D^k v^{(1)} + \mathcal{V}^k \right) \left\{ D_k \left(\mathcal{P}^{(1)} - \mathcal{P}^{(1)} \right) - \partial_\eta \left(D_k v^{(1)} + \mathcal{V}_k \right) \right\} \right. \\
 &\quad \left. + \chi^{ik} \left\{ D_i \left(D_k v^{(1)} + \mathcal{V}_k - \nu_k \right) + \frac{1}{2} \partial_\eta \chi_{ik} \right\} \right] (\epsilon + p)
 \end{aligned}$$

$$-2 \left(D^i \overset{(1)}{v} + \overset{(1)}{\mathcal{V}}^i - \overset{(1)}{\nu}^i \right) D_i \overset{(1)}{\mathcal{E}} + 2 \left(3\partial_\eta \overset{(1)}{\Psi} - \Delta \overset{(1)}{v} \right) \left(\overset{(1)}{\mathcal{E}} + \overset{(1)}{\mathcal{P}} \right). \quad (4.10)$$

4.1.3. Perturbation of the Euler equation

Here, we consider the explicit expressions of the perturbations (3.29) and (3.32) of the Euler equation for the perfect fluid.

First, we consider the background Euler equation (3.24) in the cosmological situation. In this case, the integral curve of the four-velocity $u^a = g^{ab}u_a$, whose component is given by Eq. (A.12), is a geodesic. Then, we obtain

$$a_b = 0. \quad (4.11)$$

Through the background Euler equation (3.24), this yields

$$q_b^a \nabla_a p = 0. \quad (4.12)$$

This is supported by the homogeneous spatial distribution of the background pressure. Therefore, the background Euler equation (3.24) is trivial due to the facts that the matter distribution is spatially homogeneous.

Next, we consider the components of the gauge-invariant first-order perturbation (3.29) of the Euler equation. Since the background fluid four-velocity u_a is tangent to geodesics, i.e., $a_b = 0$, we can easily see that the η -component $u^{b(1)}\mathcal{C}_b^{(p)}$ of (3.28) is trivial due to the equations $u^b q_{bc} = 0$, Eqs. (A.17), (A.59), and (4.12). On the other hand, the spatial component (i -component) of Eq. (3.28) gives the Euler equation for the perfect fluid:

$$\begin{aligned} {}^{(1)}\mathcal{C}_i^{(p)} = (\epsilon + p) \left\{ (\partial_\eta + \mathcal{H}) \left(D_i \overset{(1)}{v} + \overset{(1)}{\mathcal{V}}_i \right) + D_i \overset{(1)}{\Phi} \right\} \\ + D_i \overset{(1)}{\mathcal{P}} + \partial_\eta p \left(D_i \overset{(1)}{v} + \overset{(1)}{\mathcal{V}}_i \right) = 0, \end{aligned} \quad (4.13)$$

where we have used Eqs. (2.13), (A.17), and (A.60). This equation can be decomposed into scalar- and vector-parts

$$(\epsilon + p) \left\{ (\partial_\eta + \mathcal{H}) D_i \overset{(1)}{v} + D_i \overset{(1)}{\Phi} \right\} + D_i \overset{(1)}{\mathcal{P}} + \partial_\eta p D_i \overset{(1)}{v} = 0, \quad (4.14)$$

$$(\epsilon + p) (\partial_\eta + \mathcal{H}) \overset{(1)}{\mathcal{V}}_i + \partial_\eta p \overset{(1)}{\mathcal{V}}_i = 0. \quad (4.15)$$

Since the background value of the acceleration a_b vanishes, the η -component of ${}^{(2)}\mathcal{C}_b^{(p)}$ in Eq. (3.31) is given by

$${}^{(2)}\mathcal{C}_\eta^{(p)} = 2 {}^{(1)}\mathcal{C}_i^{(p)} \left(\overset{(1)}{\nu}^i - D^i \overset{(1)}{v} - \overset{(1)}{\mathcal{V}}^i \right) = 0, \quad (4.16)$$

where we have used Eqs. (A.17), (A.19), (A.59), (A.61), $q_{\eta c} = 0$, $p = p(\eta)$, and (4.13). On the other hand, through Eq. (2.13), (A.20), (A.21), (A.60), and (A.62),

the spatial component of the second-order perturbation (3.31) of the Euler equation is given by

$$\begin{aligned}
 {}^{(2)}\mathcal{C}_i^{(p)} &= (\epsilon + p) \left\{ (\partial_\eta + \mathcal{H}) \left(D_i {}^{(2)}v + {}^{(2)}\mathcal{V}_i \right) + D_i {}^{(2)}\Phi \right\} \\
 &\quad + D_i {}^{(2)}\mathcal{P} + \partial_\eta p \left(D_i {}^{(2)}v + {}^{(2)}\mathcal{V}_i \right) - \Xi_i^{(p)} \\
 &= 0,
 \end{aligned} \tag{4.17}$$

where $\Xi_i^{(p)}$ is the collection of the quadratic terms of the linear-order perturbations defined by

$$\begin{aligned}
 \Xi_i^{(p)} &:= -2 {}^{(1)}\Phi D_i \left\{ {}^{(1)}\mathcal{P} - (\epsilon + p) {}^{(1)}\Phi \right\} \\
 &\quad - 2(\epsilon + p) \left(\nu^j - D^j {}^{(1)}v - {}^{(1)}\mathcal{V}^j \right) \left\{ D_i {}^{(1)}\nu_j - D_j \left(D_i {}^{(1)}v + {}^{(1)}\mathcal{V}_i \right) \right\} \\
 &\quad - 2 \left({}^{(1)}\mathcal{E} + {}^{(1)}\mathcal{P} \right) \left\{ D_i {}^{(1)}\Phi + \partial_\eta \left(D_i {}^{(1)}v + {}^{(1)}\mathcal{V}_i \right) + \mathcal{H} \left(D_i {}^{(1)}v + {}^{(1)}\mathcal{V}_i \right) \right\} \\
 &\quad - 2 \left(D_i {}^{(1)}v + {}^{(1)}\mathcal{V}_i \right) \partial_\eta {}^{(1)}\mathcal{P}.
 \end{aligned} \tag{4.18}$$

As in the case of the first-order perturbations of the Euler equation, the equation (4.17) is decomposed into the scalar- and the vector-parts as

$$(\epsilon + p) \left\{ (\partial_\eta + \mathcal{H}) D_i {}^{(2)}v + D_i {}^{(2)}\Phi \right\} + D_i {}^{(2)}\mathcal{P} + \partial_\eta p D_i {}^{(2)}v = D_i \Delta^{-1} D^j \Xi_j^{(p)}, \tag{4.19}$$

$$(\epsilon + p) (\partial_\eta + \mathcal{H}) \mathcal{V}_i + \partial_\eta p \mathcal{V}_i = \Xi_i^{(p)} - D_i \Delta^{-1} D^j \Xi_j^{(p)}. \tag{4.20}$$

4.2. Imperfect fluid

4.2.1. Energy momentum tensor

Here, we consider the components of the each order perturbation of the energy momentum tensor for an imperfect fluid in the context of cosmological perturbations. In the context of the cosmological perturbations, the background spacetime is homogeneous and isotropic and the energy momentum tensor on the background spacetime is described by a perfect fluid. In other words, we can choose the four-velocity u_a of the dominant fluid so that the energy flux of the fluid vanishes,

$$q_a = 0. \tag{4.21}$$

Further, at the background level, we may neglect the anisotropic stress

$$\pi_{ab} = 0. \tag{4.22}$$

Then, the background energy momentum tensor is given by (3.2). Choosing the coordinate system as Eqs. (A.12), the components of the projection operator $q_a{}^b := g^{bc}q_{ac}$ is given through Eq. (A.16) and the background energy momentum tensor is given by Eq. (3.2).

Next, we consider the components of the gauge-invariant part of the first-order perturbation of the energy momentum tensor for imperfect fluid. In the situation where Eqs. (4.21) and (4.22) are satisfied, the properties (3.43) and (3.44) give the η -component of the first- and the second-order perturbations of the energy flux are given by

$$\hat{\mathcal{Q}}_\eta^{(1)} = 0, \quad \hat{\mathcal{Q}}_\eta^{(2)} = -2 \hat{\mathcal{Q}}_i^{(1)} \left(D^i \overset{(1)}{v} + \overset{(1)}{\mathcal{V}}^i - \overset{(1)}{\nu}^i \right), \quad (4.23)$$

where we defined $\hat{\mathcal{Q}}_a^{(1)} := a \hat{\mathcal{Q}}_a^{(1)}$, $\hat{\mathcal{Q}}_a^{(2)} := a \hat{\mathcal{Q}}_a^{(2)}$, and used Eqs. (2.13) and (A.13). In the same situation, Eq. (3.46) and (3.47) are given by

$$\hat{\Pi}_{\eta\eta}^{(1)} = 0 = \hat{\Pi}_{i\eta}^{(1)}, \quad \hat{\Pi}_{\eta\eta}^{(2)} = 0, \quad \hat{\Pi}_{\eta i}^{(2)} = -2 \hat{\Pi}_{ji}^{(1)} \left(D^j \overset{(1)}{v} + \overset{(1)}{\mathcal{V}}^j - \overset{(1)}{\nu}^j \right) \quad (4.24)$$

through Eqs. (2.13) and (A.13), where we defined $\hat{\Pi}_{ab}^{(1)} := a^2 \hat{\Pi}_{ab}^{(1)}$ and $\hat{\Pi}_{ab}^{(2)} := a^2 \hat{\Pi}_{ab}^{(2)}$. Further, the traceless properties (3.41) of the gauge-invariant parts of the first- and the second-order perturbations of the anisotropic stress are given by

$$\gamma^{ji} \hat{\Pi}_{ji}^{(1)} = 0, \quad \gamma^{ji} \hat{\Pi}_{ji}^{(2)} = 2 \chi^{ki} \hat{\Pi}_{ki}^{(1)}. \quad (4.25)$$

The components of the gauge-invariant part (3.61) of the first-order perturbation of the total energy momentum tensor for an imperfect fluid are summarized as

$$\begin{aligned} \mathcal{T}_\eta^{(1)} &= -\mathcal{E}^{(1)}, \quad \mathcal{T}_i^{(1)} = (\epsilon + p) \left(D_i \overset{(1)}{v} + \overset{(1)}{\mathcal{V}}_i \right) + \hat{\mathcal{Q}}_i^{(1)}, \\ \mathcal{T}_\eta^{(2)} &= -(\epsilon + p) \left(D^i \overset{(1)}{v} + \overset{(1)}{\mathcal{V}}^i - \overset{(1)}{\nu}^i \right) - \hat{\mathcal{Q}}^i^{(1)}, \quad \mathcal{T}_i^{(2)} = \overset{(1)}{\mathcal{P}} \gamma_i{}^j + \hat{\Pi}_i{}^j^{(1)}, \end{aligned} \quad (4.26)$$

where we have defined $\hat{\mathcal{Q}}^i := \gamma^{ij} \hat{\mathcal{Q}}_j^{(1)}$ and $\hat{\Pi}_i{}^j := \gamma^{jk} \hat{\Pi}_{ik}^{(1)}$. On the other hand, in the same situation, the components of the gauge-invariant part (3.63) of the second-order perturbation of the total energy momentum tensor for an imperfect fluid are summarized as

$$\begin{aligned} \mathcal{T}_\eta^{(2)} &= -\mathcal{E}^{(2)} - 2(\epsilon + p) \left(D^i \overset{(1)}{v} + \overset{(1)}{\mathcal{V}}^i \right) \left(D_i \overset{(1)}{v} + \overset{(1)}{\mathcal{V}}_i - \overset{(1)}{\nu}_i \right) \\ &\quad - 2 \hat{\mathcal{Q}}_i^{(1)} \left(2D^i \overset{(1)}{v} + 2\overset{(1)}{\mathcal{V}}^i - \overset{(1)}{\nu}^i \right), \end{aligned} \quad (4.27)$$

$$\begin{aligned} \mathcal{T}_i^{(2)\eta} = & 2 \left(\mathcal{E}^{(1)} + \mathcal{P}^{(1)} \right) \left(D_i^{(1)} v + \mathcal{V}_i^{(1)} \right) + (\epsilon + p) \left\{ D_i^{(2)} v + \mathcal{V}_i^{(2)} - 2 \mathcal{P}^{(1)} \left(D_i^{(1)} v + \mathcal{V}_i^{(1)} \right) \right\} \\ & + \hat{\mathcal{Q}}_i^{(2)} - 2 \mathcal{P}^{(1)} \hat{\mathcal{Q}}_i^{(1)} + 2 \hat{\Pi}_{ij}^{(1)} \left(D^j v + \mathcal{V}^j \right), \end{aligned} \quad (4.28)$$

$$\begin{aligned} \mathcal{T}_\eta^{(2)i} = & 2 \left(\mathcal{E}^{(1)} + \mathcal{P}^{(1)} \right) \left(\nu^i - D^i v - \mathcal{V}^i \right) \\ & + (\epsilon + p) \left\{ -D^i v - \mathcal{V}^i + \nu^i - 2 \mathcal{P}^{(1)} \left(D^i v + \mathcal{V}^i \right) \right. \\ & \quad \left. + 2 \left(-2 \mathcal{P}^{(1)} \gamma^{il} + \chi^{il} \right) \left(D_l v + \mathcal{V}_l - \nu_l \right) \right\} \\ & - \hat{\mathcal{Q}}^i - 2 \left(\mathcal{P}^{(1)} + 2 \mathcal{P}^{(1)} \right) \hat{\mathcal{Q}}^i + 2 \chi^{il} \hat{\mathcal{Q}}_l - 2 \hat{\Pi}^{ki} \left(D_k v + \mathcal{V}_k - \nu_k \right), \end{aligned} \quad (4.29)$$

$$\begin{aligned} \mathcal{T}_i^{(2)j} = & 2 (\epsilon + p) \left(D_i v + \mathcal{V}_i \right) \left(D^j v + \mathcal{V}^j - \nu^j \right) + \mathcal{P}^{(2)} \delta_i^j \\ & + 2 \hat{\mathcal{Q}}_i \left(D^j v + \mathcal{V}^j - \nu^j \right) + 2 \hat{\mathcal{Q}}^j \left(D_i v + \mathcal{V}_i \right) \\ & + \hat{\Pi}_i^{(2)j} + 4 \mathcal{P}^{(1)} \hat{\Pi}_i^{(1)j} - 2 \chi^{jm} \hat{\Pi}_{im}, \end{aligned} \quad (4.30)$$

where we have used Eqs. (2.13), (2.15), (4.23), (A.12)–(A.14), and defined $\hat{\mathcal{Q}}^i := \gamma^{ij} \hat{\mathcal{Q}}_j$ and $\hat{\Pi}_i^{(2)j} := \gamma^{jk} \hat{\Pi}_{ik}$.

4.2.2. Perturbative continuity equation of an imperfect fluid

Here, we derive the explicit expression of each order perturbation of the continuity equation (3.75), (3.76), and (3.77) for an imperfect fluid in terms of the components of the gauge-invariant variables.

Since we consider the situation where Eqs. (4.21) and (4.22) are satisfied, the background continuity equation coincides with Eq.(4.7) for a perfect fluid. In the same situation, the gauge-invariant expression (3.76) of the first-order perturbation of the continuity equation for an imperfect fluid is given by

$$\begin{aligned} a \left(\mathcal{C}_0^{(p)} + \mathcal{C}_0^{(i)} \right) = & \partial_\eta \mathcal{E}^{(1)} + 3\mathcal{H} \left(\mathcal{E}^{(1)} + \mathcal{P}^{(1)} \right) + (\epsilon + p) \left(\Delta v - 3\partial_\eta \mathcal{P}^{(1)} \right) \\ & + D^j \hat{\mathcal{Q}}_j = 0, \end{aligned} \quad (4.31)$$

where we have used Eqs. (A.76) and (4.23).

Next, we consider the explicit expression of the second-order perturbation (3.77) of the continuity equation for an imperfect fluid in the situation where Eqs. (4.11), (4.21), and (4.22) are satisfied. In this situation, the gauge-invariant imperfect part (3.74) of the second-order perturbation of the continuity equation is given by

$$\begin{aligned}
{}_a \mathcal{C}_0^{(i)} = & D^i \hat{\mathcal{Q}}_i + 4 \hat{\Psi}^{(1)} D^i \hat{\mathcal{Q}}_i - 2 \chi^{ik(1)} D_i \hat{\mathcal{Q}}_k + 2 \partial_\eta \hat{\mathcal{Q}}_i \left(D^i v^{(1)} + \mathcal{V}^i \right) \\
& + 2 \hat{\mathcal{Q}}_i \left\{ D^i \left(2 \Phi^{(1)} - \Psi^{(1)} \right) + 2 (\partial_\eta + 2\mathcal{H}) \left(D^i v^{(1)} + \mathcal{V}^i \right) \right\} \\
& + 2 \hat{\Pi}_{ik}^{(1)} \left(D^i \left(D^k v^{(1)} + \mathcal{V}^k - \nu^k \right) + \frac{1}{2} \partial_\eta \chi^{ik(1)} \right), \tag{4.32}
\end{aligned}$$

where we have used Eqs. (2.13), (4.23)–(4.25), (A.59), (A.60), and (A.76)–(A.79). Together with the perfect part (4.9) of the continuity equation, the explicit form of the second-order perturbation (3.77) of the continuity equation for an imperfect fluid is given by

$$\begin{aligned}
{}_a \left(\mathcal{C}_0^{(p)} + \mathcal{C}_0^{(i)} \right) = & \partial_\eta \mathcal{E}^{(2)} + 3\mathcal{H} \left(\mathcal{E}^{(2)} + \mathcal{P}^{(2)} \right) + \left(\Delta v^{(2)} - 3 \partial_\eta \Psi^{(2)} \right) (\epsilon + p) \\
& + D^i \hat{\mathcal{Q}}_i^{(2)} - \Xi_0^{(p+i)}, \tag{4.33}
\end{aligned}$$

where we defined

$$\begin{aligned}
\Xi_0^{(p+i)} := & -2 \left[\left(2 \Psi^{(1)} + \Phi^{(1)} \right) \Delta v^{(1)} + \nu^k D_k \Psi^{(1)} - 6 \Psi^{(1)} \partial_\eta \Psi^{(1)} \right. \\
& + \left(D^k v^{(1)} + \mathcal{V}^k \right) \left\{ D_k \left(\Phi^{(1)} - \Psi^{(1)} \right) + \partial_\eta \left(D_k v^{(1)} + \mathcal{V}_k \right) \right\} \\
& + \chi^{ik(1)} \left\{ D_i \left(\nu_k^{(1)} - D_k v^{(1)} - \mathcal{V}_k \right) - \frac{1}{2} \partial_\eta \chi_{ik}^{(1)} \right\} \Big] (\epsilon + p) \\
& - 2 \left(D^i v^{(1)} + \mathcal{V}^i - \nu^i \right) D_i \mathcal{E}^{(1)} - 2 \left(\Delta v^{(1)} - 3 \partial_\eta \Psi^{(1)} \right) \left(\mathcal{E}^{(1)} + \mathcal{P}^{(1)} \right) \\
& - 2 D_i \hat{\mathcal{Q}}_k^{(1)} \left(2 \Psi^{(1)} \gamma^{ik} - \chi^{ik(1)} \right) - 2 \partial_\eta \hat{\mathcal{Q}}_i^{(1)} \left(D^i v^{(1)} + \mathcal{V}^i \right) \\
& - 4 \hat{\mathcal{Q}}_i^{(1)} \left\{ D^i \left(\Phi^{(1)} - \frac{1}{2} \Psi^{(1)} \right) + (\partial_\eta + 2\mathcal{H}) \left(D^i v^{(1)} + \mathcal{V}^i \right) \right\} \\
& - 2 \hat{\Pi}_{ik}^{(1)} \left\{ D^i \left(D^k v^{(1)} + \mathcal{V}^k - \nu^k \right) + \frac{1}{2} \partial_\eta \chi^{ik(1)} \right\}. \tag{4.34}
\end{aligned}$$

4.2.3. Perturbations of the generalized Navier-Stokes equation

Here, we derive the explicit expression of each order perturbation (3.85), (3.88), and (3.91) of the generalized Navier-Stokes equation in terms of the components of the gauge-invariant variables.

First, in the situation where Eqs. (4.21) and (4.22) are satisfied, the background generalized Navier-Stokes equation (3.85) coincides with the background Euler equation (3.24) for a perfect fluid. Therefore, we obtain Eq. (4.11), again due to Eqs. (4.21) and (4.22). Hence, the background generalized Navier-Stokes equation (3.85) is trivial as mentioned above.

Next, we consider the first-order perturbation (3.88) of the generalized Navier-Stokes equation. As seen in §4.2.1, through Eqs. (4.11), (4.23)–(4.25), (A.16), (A.17), and (A.76), we can easily see that the η -component of the first-order perturbation (3.87) is trivial. On the other hand, in the same situation, the spatial component of Eq. (3.87) is given by

$$\begin{aligned} \mathcal{C}_i^{(p)} + \mathcal{C}_i^{(i)} &= (\epsilon + p) \left((\partial_\eta + \mathcal{H}) \left(D_i^{(1)} v + \mathcal{V}_i^{(1)} \right) + D_i^{(1)} \Phi \right) \\ &\quad + D_i^{(1)} \mathcal{P} + \partial_\eta p \left(D_i^{(1)} v + \mathcal{V}_i^{(1)} \right) + (\partial_\eta + 4\mathcal{H}) \hat{\mathcal{Q}}_i^{(1)} + D^k \hat{\Pi}_{ik}^{(1)} \quad (4.35) \\ &= 0, \quad (4.36) \end{aligned}$$

where we have used Eqs. (2.13), (4.23), (4.24), (A.12), (A.16)–(A.18), (A.60), (A.76), and (A.93). The scalar-part of the generalized Navier-Stokes equation (4.36) is given by

$$\begin{aligned} (\epsilon + p) \left\{ (\partial_\eta + \mathcal{H}) D_i^{(1)} v + D_i^{(1)} \Phi \right\} + D_i^{(1)} \mathcal{P} + \partial_\eta p D_i^{(1)} v \\ + D_i \Delta^{-1} D^j \left[(\partial_\eta + 4\mathcal{H}) \hat{\mathcal{Q}}_j^{(1)} + D^k \hat{\Pi}_{jk}^{(1)} \right] = 0. \quad (4.37) \end{aligned}$$

Subtracting Eq. (4.37) from Eq. (4.36), we obtain the vector-part of the first-order perturbation (4.36) of the generalized Navier-Stokes equation:

$$\begin{aligned} (\epsilon + p) (\partial_\eta + \mathcal{H}) \mathcal{V}_i^{(1)} + \partial_\eta p \mathcal{V}_i^{(1)} + (\partial_\eta + 4\mathcal{H}) \hat{\mathcal{Q}}_i^{(1)} + D^k \hat{\Pi}_{ik}^{(1)} \\ - D_i \Delta^{-1} D^j \left[(\partial_\eta + 4\mathcal{H}) \hat{\mathcal{Q}}_j^{(1)} + D^k \hat{\Pi}_{jk}^{(1)} \right] = 0. \quad (4.38) \end{aligned}$$

Finally, we consider the explicit expression of the second-order perturbation of the generalized Navier-Stokes equation given by (3.91). In the situation where Eqs. (4.11), (4.21) and (4.22) are satisfied, the η -component of this generalized Navier-Stokes equation (3.90) is trivial due to the i -component of the first-order perturbation (4.36). Actually, through Eqs. (4.23)–(4.25), (A.17)–(A.21), (A.59)–

(A·62), (A·77)–(A·79), and (A·93), the η -component of Eq. (3·90) is given by

$$\mathcal{C}_\eta^{(2)} + \mathcal{C}_\eta^{(i)} = 2 \left[\mathcal{C}_i^{(p)} + \mathcal{C}_i^{(i)} \right] \left(\nu^{(1)} - D^i v^{(1)} - \mathcal{V}^{(1)} \right) = 0. \quad (4·39)$$

On the other hand, the i -component of the second-order perturbation (3·90) of the generalized Navier-Stokes equation is given by

$$\begin{aligned} \mathcal{C}_i^{(2)} + \mathcal{C}_i^{(i)} &= (\epsilon + p) \left\{ (\partial_\eta + \mathcal{H}) \left(D_i v^{(2)} + \mathcal{V}_i^{(2)} \right) + D_i \Phi^{(2)} \right\} + D_i \mathcal{P}^{(2)} \\ &\quad + \partial_\eta p \left(D_i v^{(2)} + \mathcal{V}_i^{(2)} \right) + (\partial_\eta + 4\mathcal{H}) \hat{\mathcal{Q}}_i^{(2)} + D^k \hat{\Pi}_{ik}^{(2)} - \Xi_i^{(p+i)} \\ &= 0, \end{aligned} \quad (4·40)$$

where

$$\begin{aligned} \Xi_i^{(p+i)} &:= 2(\epsilon + p) \Phi^{(1)} D_i \Phi^{(1)} \\ &\quad - 2(\epsilon + p) \left(\nu^{(1)} - D^j v^{(1)} - \mathcal{V}^{(1)} \right) \left\{ D_i \nu_j^{(1)} - D_j \left(D_i v^{(1)} + \mathcal{V}_i^{(1)} \right) \right\} \\ &\quad - 2 \left(\mathcal{E}^{(1)} + \mathcal{P}^{(1)} \right) \left\{ (\partial_\eta + \mathcal{H}) \left(D_i v^{(1)} + \mathcal{V}_i^{(1)} \right) + D_i \Phi^{(1)} \right\} \\ &\quad - 2 \left(D_i v^{(1)} + \mathcal{V}_i^{(1)} \right) \partial_\eta \mathcal{P}^{(1)} - 2 \Phi^{(1)} D_i \mathcal{P}^{(1)} \\ &\quad + 2 \left(3\partial_\eta \Psi^{(1)} - \Delta v^{(1)} \right) \hat{\mathcal{Q}}_i^{(1)} + 2 \left\{ D_i \nu_j^{(1)} - D_j \left(D_i v^{(1)} + \mathcal{V}_i^{(1)} \right) \right\} \hat{\mathcal{Q}}_j^{(1)} \\ &\quad + 2 \left(\nu_m^{(1)} - D_m v^{(1)} - \mathcal{V}_m^{(1)} \right) D^m \hat{\mathcal{Q}}_i^{(1)} \\ &\quad - 2 \left\{ \left(\Phi^{(1)} + 2\Psi^{(1)} \right) \gamma^{kn} - \chi^{nk} \right\} D_n \hat{\Pi}_{ik}^{(1)} - 2 \left(D^j v^{(1)} + \mathcal{V}^{(1)} \right) \partial_\eta \hat{\Pi}_{ij}^{(1)} \\ &\quad + 2 \left\{ D^j \left(\Psi^{(1)} - \Phi^{(1)} \right) - (\partial_\eta + 4\mathcal{H}) \left(D^j v^{(1)} + \mathcal{V}^{(1)} \right) \right\} \hat{\Pi}_{ij}^{(1)} \\ &\quad + D_i \chi^{mk} \hat{\Pi}_{km}^{(1)}. \end{aligned} \quad (4·41)$$

Equation (4·40) is also decomposed into the scalar- and the vector-part in the similar form to Eqs. (4·37) and (4·38) with an additional source term $-\Xi_i^{(p+i)}$.

4.3. Scalar field

In the inflationary scenario of the very early universe, scalar fields play important roles which drive the inflation itself and generate the seed of the density fluctuations through their quantum fluctuations. Keeping in our mind the applications of our framework of the second-order perturbation theory to this inflationary scenario, in this subsection, we summarize the explicit expressions of the perturbative the energy momentum tensor and the Klein-Gordon equation.

4.3.1. Energy momentum tensor

Here, we derive the explicit expression of the components of the energy momentum tensors (3.93), (3.96), (3.97) for a single scalar field with the potential $V(\phi)$ in gauge-invariant form.

In the cosmological situation, we consider the homogeneous background field:

$$\varphi = \varphi(\eta). \quad (4.42)$$

Through the background metric (2.12), the components of the background the energy momentum tensor (3.93) are given by

$$T_{\eta}^{\eta} = -\frac{1}{2a^2}(\partial_{\eta}\varphi)^2 - V(\bar{\varphi}), \quad T_{\eta}^i = 0 = T_i^{\eta}, \quad (4.43)$$

$$T_i^j = \frac{1}{2a^2} \{ (\partial_{\eta}\varphi)^2 - 2a^2 V(\bar{\varphi}) \} \gamma_i^j. \quad (4.44)$$

Through the components (2.13) of the gauge-invariant part of the first-order metric perturbation, the components of the first-order perturbation of the energy-momentum tensor of a single scalar field are given by

$$^{(1)}\mathcal{T}_{\eta}^{\eta} = -\frac{1}{a^2} \left\{ \partial_{\eta}\varphi \partial_{\eta}\varphi_1 - \overset{(1)}{\Phi} (\partial_{\eta}\varphi)^2 + a^2 \frac{\partial V}{\partial \varphi} \varphi_1 \right\}, \quad (4.45)$$

$$^{(1)}\mathcal{T}_{\eta}^i = \frac{1}{a^2} \partial_{\eta}\varphi \left(D^i \varphi_1 + \overset{(1)}{\nu}^i \partial_{\eta}\varphi \right), \quad (4.46)$$

$$^{(1)}\mathcal{T}_i^{\eta} = -\frac{1}{a^2} D_i \varphi_1 \partial_{\eta}\varphi, \quad (4.47)$$

$$^{(1)}\mathcal{T}_i^j = \frac{1}{a^2} \gamma_i^j \left\{ \partial_{\eta}\varphi \partial_{\eta}\varphi_1 - \overset{(1)}{\Phi} (\partial_{\eta}\varphi)^2 - a^2 \frac{\partial V}{\partial \varphi} \varphi_1 \right\}. \quad (4.48)$$

Finally, we consider the gauge-invariant part (3.97) of the second-order perturbation of the energy momentum tensor for a scalar field. Through the components (2.13) and (2.15) of the gauge-invariant parts of the first- and the second-order metric perturbations and the homogeneous background condition (4.42), the components of the second-order perturbation of the energy-momentum tensor for a single scalar field are given by

$$^{(2)}\mathcal{T}_{\eta}^{\eta} = -\frac{1}{a^2} \left\{ \partial_{\eta}\varphi \partial_{\eta}\varphi_2 - (\partial_{\eta}\varphi)^2 \overset{(2)}{\Phi} + a^2 \varphi_2 \frac{\partial V}{\partial \varphi} - 4 \partial_{\eta}\varphi \overset{(1)}{\Phi} \partial_{\eta}\varphi_1 + 4 (\partial_{\eta}\varphi)^2 (\overset{(1)}{\Phi})^2 \right\}$$

$$-(\partial_\eta \varphi)^2 \nu^i \nu_i^{(1)} + (\partial_\eta \varphi_1)^2 + D_i \varphi_1 D^i \varphi_1 + a^2 (\varphi_1)^2 \frac{\partial^2 V}{\partial \varphi^2} \Big\}, \quad (4.49)$$

$${}^{(2)}\mathcal{T}_i{}^\eta = -\frac{1}{a^2} \left\{ \partial_\eta \varphi \left(D_i \varphi_2 - 4 D_i \varphi_1 \overset{(1)}{\Phi} \right) + 2 D_i \varphi_1 \partial_\eta \varphi_1 \right\}, \quad (4.50)$$

$$\begin{aligned} {}^{(2)}\mathcal{T}_\eta{}^i &= \frac{1}{a^2} \left[\partial_\eta \varphi D^i \varphi_2 + 2 \partial_\eta \varphi_1 D^i \varphi_1 + 2 \partial_\eta \varphi \left(2 \nu^i \partial_\eta \varphi_1 + 2 \overset{(1)}{\Psi} D^i \varphi_1 - \chi^{il} \overset{(1)}{D}_l \varphi_1 \right) \right. \\ &\quad \left. + (\partial_\eta \varphi)^2 \left(\nu^i - 4 \overset{(1)}{\Phi} \nu^i + 4 \overset{(1)}{\Psi} \nu^i - 2 \chi^{ik} \overset{(1)}{\nu}_k \right) \right], \end{aligned} \quad (4.51)$$

$$\begin{aligned} {}^{(2)}\mathcal{T}_i{}^j &= \frac{2}{a^2} \left[D_i \varphi_1 D^j \varphi_1 + D_i \varphi_1 \overset{(1)}{\nu}^j \partial_\eta \varphi \right. \\ &\quad \left. + \frac{1}{2} \gamma_i{}^j \left\{ + \partial_\eta \varphi \left(\partial_\eta \varphi_2 - 4 \overset{(1)}{\Phi} \partial_\eta \varphi_1 - 2 \overset{(1)}{\nu}_l D^l \varphi_1 \right) \right. \right. \\ &\quad \left. + (\partial_\eta \varphi)^2 \left(4 \left(\overset{(1)}{\Phi} \right)^2 - \overset{(1)}{\nu}^l \overset{(1)}{\nu}_l - \overset{(2)}{\Phi} \right) + (\partial_\eta \varphi_1)^2 - D_l \varphi_1 D^l \varphi_1 \right. \\ &\quad \left. \left. - a^2 \varphi_2 \frac{\partial V}{\partial \varphi} - a^2 (\varphi_1)^2 \frac{\partial^2 V}{\partial \varphi^2} \right\} \right]. \end{aligned} \quad (4.52)$$

In the components (4.49)–(4.52), the vector- and the tensor-modes of the first-order perturbations are included into our consideration, which are ignored in KN2007.

4.3.2. Perturbative Klein-Gordon equations

Here, we consider the explicit expression of the gauge-invariant part of each order perturbation (3.100) and (3.107) of the Klein-Gordon equation in the context of cosmological perturbations. Since the background field φ in cosmology is homogeneous and the background metric is given by (2.12), the background Klein-Gordon equation (3.100) yields

$$-a^2 C_{(K)} = \partial_\eta^2 \varphi + 2\mathcal{H} \partial_\eta \varphi + a^2 \frac{\partial V}{\partial \varphi}(\varphi) = 0. \quad (4.53)$$

On the other hand, the gauge-invariant part $\overset{(1)}{\mathcal{C}}_{(K)}$ of the first-order perturbation of $\bar{C}_{(K)}$, which is defined by Eq. (3.105), is explicitly given by

$$\begin{aligned} -a^2 \overset{(1)}{\mathcal{C}}_{(K)} &= \partial_\eta^2 \varphi_1 + 2\mathcal{H} \partial_\eta \varphi_1 - \Delta \varphi_1 - \left(\partial_\eta \overset{(1)}{\Phi} + 3 \partial_\eta \overset{(1)}{\Psi} \right) \partial_\eta \varphi \\ &\quad + 2a^2 \overset{(1)}{\Phi} \frac{\partial V}{\partial \varphi}(\varphi) + a^2 \varphi_1 \frac{\partial^2 V}{\partial \varphi^2}(\varphi) = 0, \end{aligned} \quad (4.54)$$

where we have used the background metric (2.12), the components (2.13) of the gauge-invariant part of the linear-order metric perturbation, the homogeneous condition (4.42) for the background field φ , and the background Klein-Gordon equation (4.53).

Finally, we consider the explicit expression of the gauge-invariant second-order perturbation of the Klein-Gordon equation in Eqs. (3.107). The gauge-invariant expression of $\mathcal{C}_{(K)}^{(2)}$ for the Klein-Gordon equation is defined by (3.106). Through Eqs. (2.13), (2.15), (4.42), and the background Klein-Gordon equation (4.53) and its first-order perturbation (4.54), the explicit expression of $\mathcal{C}_{(K)}^{(2)}$ is given by

$$\begin{aligned} -a^2 \mathcal{C}_{(K)}^{(2)} &= \partial_\eta^2 \varphi_2 + 2\mathcal{H} \partial_\eta \varphi_2 - \Delta \varphi_2 - \left(\partial_\eta \overset{(2)}{\Phi} + 3\partial_\eta \overset{(2)}{\Psi} \right) \partial_\eta \varphi \\ &\quad + 2a^2 \overset{(2)}{\Phi} \frac{\partial V}{\partial \bar{\varphi}}(\varphi) + a^2 \varphi_2 \frac{\partial^2 V}{\partial \bar{\varphi}^2}(\varphi) - \Xi_{(K)} = 0, \end{aligned} \quad (4.55)$$

where we defined

$$\begin{aligned} \Xi_{(K)} &:= 2\partial_\eta \left(\overset{(1)}{\Phi} + 3\overset{(1)}{\Psi} \right) \partial_\eta \varphi_1 + 2D^i \left(\overset{(1)}{\Phi} - \overset{(1)}{\Psi} \right) D_i \varphi_1 + 4 \left(\overset{(1)}{\Phi} + \overset{(1)}{\Psi} \right) \Delta \varphi_1 \\ &\quad - 4a^2 \overset{(1)}{\Phi} \varphi_1 \frac{\partial^2 V}{\partial \bar{\varphi}^2}(\varphi) - a^2 (\varphi_1)^2 \frac{\partial^3 V}{\partial \bar{\varphi}^3}(\varphi) \\ &\quad + 2(\partial_\eta + 2\mathcal{H}) \overset{(1)}{\nu}^i D_i \varphi_1 + 4 \overset{(1)}{\nu}^i \partial_\eta D_i \varphi_1 - 2 \overset{(1)}{\chi}^{ij} D_j D_i \varphi_1 \\ &\quad + 2 \left\{ -2 \overset{(1)}{\Phi} \partial_\eta \overset{(1)}{\Phi} + 6 \overset{(1)}{\Psi} \partial_\eta \overset{(1)}{\Psi} - D^i \left(\overset{(1)}{\Phi} + \overset{(1)}{\Psi} \right) \overset{(1)}{\nu}_i \right. \\ &\quad \left. + \overset{(1)}{\nu}_i \partial_\eta \overset{(1)}{\nu}^i - \overset{(1)}{\chi}^{ij} \left(D_i \overset{(1)}{\nu}_j - \frac{1}{2} \partial_\eta \overset{(1)}{\chi}_{ij} \right) \right\} \partial_\eta \varphi \\ &\quad - 2a^2 \overset{(1)}{\nu}^i \overset{(1)}{\nu}_i \frac{\partial V}{\partial \bar{\varphi}}(\varphi). \end{aligned} \quad (4.56)$$

§5. Summary and Discussions

In summary, we have derived the explicit expressions of the second-order perturbations of the energy momentum tensor for a perfect fluid, an imperfect fluid, and a scalar field. Further, we also derived the explicit expression of the second-order perturbations the continuity equation and the Euler equation for a perfect fluid, the continuity equation and the generalized Navier-Stokes equation for an imperfect fluid, and the Klein-Gordon equation for a scalar field. As in KN2007,⁵⁾ we have again confirmed that the general formulation of the second-order perturbation theory developed in the papers KN2003⁹⁾ and KN2005¹⁰⁾ does work and it is applicable to the perturbations of the energy momentum tensors and the equations of motion for matter fields in the cosmological perturbation theory. In the derivations of these equations, we have again seen that the decomposition formulae (2.20) and (2.21) of the perturbations for the arbitrary fields play crucial roles in the gauge-invariant perturbation theory. Since the general relativistic higher-order perturbation theory requires the long algebraic calculations, in many cases, it is difficult to have confidence in the resulting long equations. In spite of this fact, we showed that all

perturbative variables are decomposed into the gauge-invariant and gauge-variant variables in the forms (2.20) and (2.21). This implies that the decomposition formulae (2.20) and (2.21) are useful to check whether the resulting equations are correct or not. This is the main point of this paper.

As mentioned in Introduction (§1), there are some attempts of the derivations of the perturbative expressions of the evolution equations of matter fields to the second-order.¹⁶⁾ For example, Noh and Hwang¹⁶⁾ also summarized the formulae of the energy momentum tensor, equations of motion for an imperfect fluid in the cosmological situation up to second order without any gauge fixing. However, they implicitly imposed so-called “normal frame condition” in their formulae. In our notation, this normal frame condition corresponds to $D_i \overset{(1)}{v} + \overset{(1)}{\mathcal{V}}_i = D_i \overset{(2)}{v} + \overset{(2)}{\mathcal{V}}_i = 0$. Due to this condition, their formulae are not equivalent to ours. Further, their formulae include gauge degree of freedom and the resulting formulae have some complicated forms. On the other hand, in this paper, all formulae are given in terms of gauge-invariant variables and there is no gauge ambiguity in these expressions. In this sense, the formulae in this paper are irreducible. We also have to emphasize that we did not fix any gauge when we derive any perturbative formulae in this paper. The gauge-invariant treatments of perturbations are equivalent to the complete gauge-fixing method. Therefore, we have realized the complete gauge-fixing without any gauge fixing. This is an advantage of the gauge-invariant perturbation theory in this paper.

As emphasized in KN2007, the key point of our procedure is in the assumption which state that we already know the procedure to decompose the first-order metric perturbation into the gauge-invariant and variant parts. Mathematically speaking, this assumption is expressed by the statement that *there is a vector field X^a which is constructed from some components of the linear-order metric perturbation so that its gauge transformation rule is given by the second equation in Eqs. (2.9) and the linear-order metric perturbation is decomposed as Eq. (2.8)*. As shown in KN2007 this assumption is correct at least in the cosmological perturbation case. However, even in the cosmological perturbation case, homogeneous modes of perturbations are excluded from our consideration because we assumed the existence of the Green functions Δ^{-1} , $(\Delta + 2K)^{-1}$, and $(\Delta + 3K)^{-1}$. These homogeneous modes of the cosmological perturbations will be some dynamical degrees of freedom in compact universes and we cannot say that these modes are unphysical. If we want to include these homogeneous modes into our considerations, separate treatments will be required. Besides this detailed problem to extend the domain of functions for perturbations to includes homogeneous modes, the above assumption is correct on some background spacetime other than the cosmological background spacetime.¹¹⁾ Therefore, we propose a conjecture that the above assumption is correct on any background spacetime. To clarify whether this conjecture is true or not, non-local arguments on the background spacetime will be necessary, since gauge-invariant variables are non-local one by their definitions.

Even if the assumption is correct on any background spacetime, the other problem is in the interpretations of the gauge-invariant variables. We have commented

on the non-uniqueness in the definitions of the gauge-invariant variables in §2.2. This non-uniqueness in the definition of gauge-invariant variables also leads some ambiguities in the interpretations of gauge-invariant variables. Although the precise interpretations of the gauge-invariant variables will be accomplished by the clarification of the relations between gauge-invariant variables and observables in experiments and observations, some geometrical interpretations are given through the explicit expressions of the gauge-invariant variables for the perturbations of geometrical quantities like the acceleration \bar{a}_a , the expansion $\bar{\theta}$, the shear $\bar{\sigma}_{ab}$, and the rotation $\bar{\omega}_{ab}$. For example, it is well-known that the expressions (A·59) and (A·60) of the first-order perturbation of the acceleration vector give the clear interpretation of the scalar-mode $\bar{\Phi}^{(1)}$ of the first-order metric perturbation. If \bar{u}^a is a tangent to a geodesic even in the first-order perturbation, Eq. (A·60) gives the equation of motion of a free-falling object in expanding universe and we can regard the scalar function $\bar{\Phi}^{(1)}$ as the Newton gravitational potential. Further, the similar interpretation for $\bar{\Phi}^{(2)}$ is also possible through Eq. (A·62) though the quadratic terms of the linear-order perturbations in Eq. (A·62) are difficult to interpret. Similarly, $\bar{\Psi}^{(1)}$ and $\bar{\Psi}^{(2)}$ contribute to the expansion of the first- and the second-order perturbations of the expansion as in Eqs. (A·94) and (A·95), respectively, and the vector- and tensor-modes $\bar{\nu}_i^{(1)}$, $\bar{\nu}_i^{(2)}$, $\bar{\chi}_{ij}^{(1)}$, and $\bar{\chi}_{ij}^{(2)}$ contribute to the perturbations of the shear tensor through Eq. (A·106) and (A·109). As seen in these equations, the contributions of the second-order gauge-invariant variables $\bar{\Phi}^{(2)}$, $\bar{\Psi}^{(2)}$, $\bar{\nu}_i^{(2)}$, and $\bar{\chi}_{ij}^{(2)}$ to the gauge-invariant part of the second-order perturbations of the geometrical quantities $\bar{\theta}$, $\bar{\sigma}_{ab}$, $\bar{\omega}_{ab}$ are similar to the contribution of the first-order gauge-invariant variables $\bar{\Phi}^{(1)}$, $\bar{\Psi}^{(1)}$, $\bar{\nu}_i^{(1)}$, and $\bar{\chi}_{ij}^{(1)}$ of the first-order metric perturbation to the gauge-invariant part of the first-order perturbations of these geometrical quantities. Therefore, we may regard the gauge-invariant variables $\bar{\Phi}^{(2)}$, $\bar{\Psi}^{(2)}$, $\bar{\nu}_i^{(2)}$, and $\bar{\chi}_{ij}^{(2)}$ for the second-order metric perturbation are natural extensions of the gauge-invariant variables $\bar{\Phi}^{(1)}$, $\bar{\Psi}^{(1)}$, $\bar{\nu}_i^{(1)}$, and $\bar{\chi}_{ij}^{(1)}$ for the first-order perturbations to the second-order one and the geometrical interpretations of these second-order variables will be similar to those of the first-order one.

In all derivations of the perturbative expressions of the energy momentum tensors and the equations of motion in §3, we did not use any information of the Einstein equations nor equations of state for the matter fields. This implies that the formulae in §3 are valid for the very wide class of perturbation theories of gravitational field. However, we have to emphasize that the equations of motion derived in §4 are not able to solve by themselves. These equations include metric perturbations. Therefore, to solve these equations, we have to use the Einstein equations. Further, we have treated the anisotropic stress and the energy flux in the case of an imperfect fluid as independent variables of other perturbative variables. If we specify the micro-physics, these variables are related to the other perturbative variables.^{7),14)} Therefore, in general, it is meaningless to try to discuss the solutions to the equations

derived in §4 by themselves. To discuss the solutions to them, we have to treat the Einstein equations and we have to specify the interactions between matter fields at micro- or macro-physical level.

Finally, we can show that these equations of motion are not independent of the perturbative Einstein equations in the case where the spacetime is filled with a single matter field.²⁰⁾ This is a natural result because the Einstein equation includes the equations of motion for the matter field through the Bianchi identity at least in the case of the single matter content. Therefore, we may concentrate on the Einstein equations when we solve the system of the single matter field. On the other hand, when we consider the multi-fluids or multi-fields system, we will have to specify the interactions between these matter fields and to treat the equations of motion derived in this paper, seriously. Therefore, in the realistic situations of cosmology, the formulae summarized in this paper will play crucial role.

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Appendix A

— Perturbations of geometrical quantities —

The explicit expressions of the perturbative equations of motion for fluids in §3 are given by the perturbative expansions of the fluid component $\bar{\epsilon}$, \bar{p} , \bar{u}_a , \bar{q}_a , and $\bar{\pi}_{ab}$ as Eqs. (2·23)–(2·25), (2·34)–(2·35) together with the perturbative expansion (2·7) of the metric perturbation. Further, to derive the perturbations of the equations of motion for matter fields, it is convenient to consider the perturbation of the tensor $\bar{\nabla}_a \bar{u}_b$. From the components of the tensor $\bar{\nabla}_a \bar{u}_b$, we define the acceleration, the expansion, the shear, and the rotation associated with the four-velocity \bar{u}_a . The acceleration \bar{a}_b are defined by

$$\bar{a}_b := \bar{u}^a \bar{\nabla}_a \bar{u}_b \quad (\text{A} \cdot 1)$$

and the expansion, the shear, and the rotation are defined as the trace, the traceless symmetric part, the antisymmetric part of the tensor

$$\bar{B}_{ab} := \bar{q}_a{}^c \bar{q}_b{}^d \bar{\nabla}_c \bar{u}_d = \bar{\nabla}_a \bar{u}_b + \bar{u}_a \bar{a}_b, \quad (\text{A} \cdot 2)$$

respectively. In this Appendix, we show the perturbations of the tensor field $\bar{\nabla}_a \bar{u}_b$ and the acceleration, expansion, shear, and rotation. All of the perturbations of these geometrical quantities are also decomposed into the gauge-invariant and the gauge-variant parts as Eqs. (2·20) and (2·21).

A.1. Components of the gauge-invariant three-metric

Before discussing the components of the perturbations of tensor \bar{B}_{ab} , we first derive the components of the perturbation of the three-metric \bar{q}_{ab} , which is defined by

$$\bar{q}_{ab} := \bar{g}_{ab} + \bar{u}_a \bar{u}_b \quad (\text{A} \cdot 3)$$

and it is expanded as

$$\bar{q}_{bc} = q_{bc} + \lambda \, {}^{(1)}(q_{bc}) + \frac{1}{2} \lambda^2 \, {}^{(2)}(q_{bc}) + O(\lambda^3). \quad (\text{A} \cdot 4)$$

From the perturbative expansions (2·7), (2·25), each order perturbation of the three metric is summarized as

$$q_{ab} := g_{ab} + u_a u_b, \quad (\text{A} \cdot 5)$$

$${}^{(1)}(q_{ab}) := h_{ab} + u_a \, {}^{(1)}(u_b) + {}^{(1)}(u_a) \, u_b, \quad (\text{A} \cdot 6)$$

$${}^{(2)}(q_{ab}) := l_{ab} + u_a \, {}^{(2)}(u_b) + 2 \, {}^{(1)}(u_a) {}^{(1)}(u_b) + {}^{(2)}(u_a) \, u_b. \quad (\text{A} \cdot 7)$$

Further, substituting the decompositions (2·8), (2·10) of the first- and the second-order metric perturbations and the definitions of the gauge-invariant variables for the first- and the second-order perturbations of the fluid four-velocity in Eqs. (2·26) and (2·28) into Eqs. (A·6) and (A·7), we can show that each order perturbation of \bar{q}_{ab} is decomposed into the gauge-invariant and gauge-variant parts as

$${}^{(1)}(q_{bc}) = \mathcal{Q}_{bc} + \mathcal{L}_X q_{bc}, \quad (\text{A} \cdot 8)$$

$${}^{(2)}(q_{bc}) = \mathcal{Q}_{bc} + 2 \mathcal{L}_X \, {}^{(1)}(q_{bc}) + \{ \mathcal{L}_Y - \mathcal{L}_X^2 \} q_{bc}. \quad (\text{A} \cdot 9)$$

Here, the gauge-invariant parts $\mathcal{Q}_{ab}^{(1)}$ and $\mathcal{Q}_{ab}^{(2)}$ are defined by

$$\mathcal{Q}_{ab}^{(1)} := \mathcal{H}_{ab} + u_a \, \mathcal{U}_b + \mathcal{U}_a \, u_b, \quad (\text{A} \cdot 10)$$

$$\mathcal{Q}_{ab}^{(2)} := \mathcal{L}_{ab} + u_a \, \mathcal{U}_b^{(2)} + \mathcal{U}_a^{(2)} \, u_b + 2 \, \mathcal{U}_a^{(1)} \mathcal{U}_b^{(1)}. \quad (\text{A} \cdot 11)$$

To derive the components of the gauge-invariant part (A·10) and (A·11) of the perturbative three-metric, the components of the gauge-invariant part of the fluid four-velocity are necessary. Components of the background and the first-order perturbations of the four-velocity are summarized as

$$u_a = -a(d\eta)_a, \quad (\text{A} \cdot 12)$$

$$\mathcal{U}_a^{(1)} = -a \mathcal{F}^{(1)} (d\eta)_a + a \left(D_i^{(1)} v^{(1)} + \mathcal{V}_i^{(1)} \right) (dx^i)_a, \quad D^i \mathcal{V}_i^{(1)} = 0, \quad (\text{A}\cdot 13)$$

$$\mathcal{U}_a^{(2)} = \mathcal{U}_\eta^{(2)} (d\eta)_a + a \left(D_i^{(2)} v^{(2)} + \mathcal{V}_i^{(2)} \right) (dx^i)_a, \quad D^i \mathcal{V}_i^{(2)} = 0, \quad (\text{A}\cdot 14)$$

where the η -component of the first-order perturbation $\mathcal{U}_a^{(1)}$ is determined by Eq. (3-9) and the η -component of the gauge-invariant part $\mathcal{U}_a^{(2)}$ in Eq. (A-14) is given by

$$\mathcal{U}_\eta^{(2)} = a \left\{ \left(\mathcal{F}^{(1)} \right)^2 - \mathcal{F}^{(2)} - \left(D_i^{(1)} v^{(1)} + \mathcal{V}_i^{(1)} - \nu_i^{(1)} \right) \left(D^i v^{(1)} + \mathcal{V}^i - \nu^i \right) \right\} \quad (\text{A}\cdot 15)$$

through Eq. (3-10), where we have used the components (2-13) and (2-15) of the first- and the second-order metric perturbations.

The components of the background three-metric q_{ab} defined by Eq. (A-5) is given by

$$q_{ab} = a^2 \gamma_{ij} (dx^i)_a (dx^j)_b =: a^2 \gamma_{ab}. \quad (\text{A}\cdot 16)$$

Through Eqs. (2-13), (A-12), and (A-13), the components of the gauge-invariant part (A-10) of the first-order perturbation of the three-metric, which is defined by (A-10), are given by

$$\mathcal{Q}_{\eta\eta}^{(1)} = 0, \quad \mathcal{Q}_{i\eta}^{(1)} = \mathcal{Q}_{\eta i}^{(1)} = a^2 \left(\nu_i^{(1)} - D_i^{(1)} v^{(1)} - \mathcal{V}_i^{(1)} \right), \quad (\text{A}\cdot 17)$$

$$\mathcal{Q}_{ij}^{(1)} = \mathcal{H}_{ij} = a^2 \left(-2 \Psi^{(1)} \gamma_{ij} + \chi_{ij}^{(1)} \right). \quad (\text{A}\cdot 18)$$

Similarly, through Eqs. (2-15), (A-12)–(A-15), the components of the gauge-invariant part of the second-order perturbation of the three-metric, which is defined by Eq. (A-11), are summarized as

$$\mathcal{Q}_{\eta\eta}^{(2)} = 2a^2 \left(D_i^{(1)} v^{(1)} + \mathcal{V}_i^{(1)} - \nu_i^{(1)} \right) \left(D^i v^{(1)} + \mathcal{V}^i - \nu^i \right), \quad (\text{A}\cdot 19)$$

$$\mathcal{Q}_{i\eta}^{(2)} = \mathcal{Q}_{\eta i}^{(2)} = a^2 \left\{ \nu_i^{(2)} - D_i^{(2)} v^{(2)} - \mathcal{V}_i^{(2)} - 2 \mathcal{F}^{(1)} \left(D_i^{(1)} v^{(1)} + \mathcal{V}_i^{(1)} \right) \right\}, \quad (\text{A}\cdot 20)$$

$$\mathcal{Q}_{ij}^{(2)} = a^2 \left\{ -2 \Psi^{(2)} \gamma_{ij} + \chi_{ij}^{(2)} + 2 \left(D_i^{(1)} v^{(1)} + \mathcal{V}_i^{(1)} \right) \left(D_j^{(1)} v^{(1)} + \mathcal{V}_j^{(1)} \right) \right\}. \quad (\text{A}\cdot 21)$$

A.2. Perturbation of the tensor $\bar{A}_{ab} = \bar{\nabla}_a \bar{u}_b$

Here, we consider the perturbation of the tensor defined by

$$\bar{A}_{ab} := \bar{\nabla}_a \bar{u}_b. \quad (\text{A}\cdot 22)$$

The covariant derivative $\bar{\nabla}_a$ associated with the metric \bar{g}_{ab} on the physical spacetime is related to the covariant derivative ∇_a associated with the background metric g_{ab} as

$$\bar{A}_{ab} = \nabla_a \bar{u}_b - C_{ba}^c \bar{u}_c, \quad (\text{A}\cdot 23)$$

where the connection C_{ba}^c is given by

$$C_{ab}^c = \frac{1}{2} \bar{g}^{cd} (\nabla_a \bar{g}_{db} + \nabla_b \bar{g}_{da} - \nabla_d \bar{g}_{ab}) \quad (\text{A}\cdot 24)$$

$$=: \lambda H_{ab}{}^c[h] + \frac{1}{2} \lambda^2 \left(H_{ab}{}^c[l] - 2h^{cd} H_{abd}[h] \right) + O(\lambda^3), \quad (\text{A}\cdot 25)$$

where we defined

$$H_{abc}[t] := \frac{1}{2} (\nabla_a t_{cb} + \nabla_b t_{ca} - \nabla_c t_{ab}), \quad H_{ab}{}^c[t] := g^{cd} H_{abd}[t] \quad (\text{A}\cdot 26)$$

for any tensor t_{ab} of the second rank. Through Eqs. (2·25) and (A·25), the tensor \bar{A}_{ab} is expanded as

$$\begin{aligned} \bar{A}_{ab} = & \nabla_a u_b + \lambda \left(\nabla_a (u_b)^{(1)} - H_{ba}{}^c[h] u_c \right) \\ & + \frac{1}{2} \lambda^2 \left\{ \nabla_a (u_b)^{(2)} - 2H_{ab}{}^c[h] (u_c)^{(1)} - \left(H_{ab}{}^c[l] - 2h^{cd} H_{abd}[h] \right) u_c \right\} \\ & + O(\lambda^3) \end{aligned} \quad (\text{A}\cdot 27)$$

$$=: A_{ab} + \lambda A_{ab}^{(1)} + \frac{1}{2} \lambda^2 A_{ab}^{(2)} + O(\lambda^3). \quad (\text{A}\cdot 28)$$

Further, through Eqs. (2·8) and (2·10), the first- and the second-order perturbations of the connection (A·25) are given by

$$H_{ab}{}^c[h] = H_{ab}{}^c[\mathcal{H}] + \nabla_a \nabla_b X^c + R_{aeb}{}^c X^e, \quad (\text{A}\cdot 29)$$

$$\begin{aligned} H_{ab}{}^c[l] - 2h^{cd} H_{abd}[h] = & H_{ab}{}^c[\mathcal{L}] - 2\mathcal{H}^{cd} H_{abd}[\mathcal{H}] \\ & + 2\mathcal{L}_X H_{ab}{}^c[h] + \nabla_a \nabla_b Y^c + Y^e R_{aeb}{}^c \\ & - \mathcal{L}_X (\nabla_a \nabla_b X^c + R_{aeb}{}^c X^e). \end{aligned} \quad (\text{A}\cdot 30)$$

The components of the tensor $H_{ab}{}^c[\mathcal{H}]$ are summarized in Appendix of KN2007 and the components of $H_{ab}{}^c[\mathcal{L}]$ can be derived from the components of $H_{ab}{}^c[\mathcal{H}]$ by the replacements

$$\begin{aligned} \overset{(1)}{\Phi} \rightarrow \overset{(2)}{\Phi}, \quad \overset{(1)}{\nu}_i \rightarrow \overset{(2)}{\nu}_i, \quad \overset{(1)}{\Psi} \rightarrow \overset{(2)}{\Psi}, \quad \overset{(1)}{\chi}_{ij} \rightarrow \overset{(2)}{\chi}_{ij}. \end{aligned} \quad (\text{A}\cdot 31)$$

Then, through the last equation in Eqs. (2·26) and (A·29), the first-order perturbation of the tensor \bar{A}_{ab} in Eq. (A·28) is decomposed as

$$A_{ab} = A_{ab}^{(1)} + \mathcal{L}_X A_{ab}, \quad (\text{A}\cdot 32)$$

where

$$\mathcal{A}_{ab}^{(1)} := \nabla_a \mathcal{U}_b^{(1)} - u_c H_{ab}{}^c [\mathcal{H}]. \quad (\text{A}\cdot 33)$$

$\mathcal{A}_{ab}^{(1)}$ in Eq. (A·33) is the gauge-invariant part of the first-order perturbation $\bar{A}_{ab}^{(1)}$ of the tensor \bar{A}_{ab} and we have verified that the first-order perturbation $\bar{A}_{ab}^{(1)}$ is decomposed as Eq. (2·20). Similarly, through the last equation in Eqs. (2·26), (2·28), (A·29), and (A·30), the second-order perturbation $\bar{A}_{ab}^{(2)}$ in Eq. (A·28) of \bar{A}_{ab} is decomposed as

$$\bar{A}_{ab}^{(2)} = \mathcal{A}_{ab}^{(2)} + \mathcal{L}_Y A_{ab} - \mathcal{L}_X^2 A_{ab} + 2\mathcal{L}_X \bar{A}_{ab}^{(1)}, \quad (\text{A}\cdot 34)$$

where

$$\mathcal{A}_{ab}^{(2)} := \nabla_a \mathcal{U}_b^{(2)} - u_c H_{ba}{}^c [\mathcal{L}] + 2u_c \mathcal{H}^{cd} H_{bad} [\mathcal{H}] - 2H_{ba}{}^c [\mathcal{H}] \mathcal{U}_c^{(1)}. \quad (\text{A}\cdot 35)$$

$\mathcal{A}_{ab}^{(2)}$ in Eq. (A·34) is the gauge-invariant part of the second-order perturbation $\bar{A}_{ab}^{(2)}$ of \bar{A}_{ab} and we have verified that the second-order perturbation $\bar{A}_{ab}^{(2)}$ is decomposed as Eq. (2·21).

Now, we consider the components of gauge-invariant parts of the perturbations of the tensor \bar{A}_{ab} . First, we consider the components of the background value A_{ab} . Through Eq. (A·12), we obtain

$$A_{ab} = a\mathcal{H}\gamma_{ij}(dx^i)_a(dx^j)_b = a\mathcal{H}\gamma_{ab}, \quad (\text{A}\cdot 36)$$

where $\mathcal{H} := \partial_\eta a/a$. The components of the gauge-invariant parts $\mathcal{A}_{ab}^{(1)}$ of the first-order perturbation of the tensor \bar{A}_{ab} are summarized as

$$\mathcal{A}_{\eta\eta}^{(1)} = 0, \quad (\text{A}\cdot 37)$$

$$\mathcal{A}_{\eta i}^{(1)} = a \left(\partial_\eta D_i \bar{v}^{(1)} + \partial_\eta \bar{\mathcal{V}}_i^{(1)} + D_i \bar{\Phi}^{(1)} + \mathcal{H} \bar{\nu}_i^{(1)} \right), \quad (\text{A}\cdot 38)$$

$$\mathcal{A}_{i\eta}^{(1)} = -a\mathcal{H} \left(D_i \bar{v}^{(1)} + \bar{\mathcal{V}}_i^{(1)} - \bar{\nu}_i^{(1)} \right), \quad (\text{A}\cdot 39)$$

$$\begin{aligned} \mathcal{A}_{ij}^{(1)} = a \left\{ D_i D_j \bar{v}^{(1)} + D_i \bar{\mathcal{V}}_j^{(1)} - D_{(i} \bar{\nu}_{j)}^{(1)} - \left(\mathcal{H} \bar{\Phi}^{(1)} + 2\mathcal{H} \bar{\Psi}^{(1)} + \partial_\eta \bar{\Psi}^{(1)} \right) \gamma_{ij} \right. \\ \left. + \mathcal{H} \bar{\chi}_{ij}^{(1)} + \frac{1}{2} \partial_\eta \bar{\chi}_{ij}^{(1)} \right\}. \end{aligned} \quad (\text{A}\cdot 40)$$

Finally, the components of the gauge-invariant part $\mathcal{A}_{ab}^{(2)}$ of the second-order perturbation of the tensor \bar{A}_{ab} are summarized as

$$\mathcal{A}_{\eta\eta}^{(2)} = 2a \left\{ \partial_\eta \left(D_i \bar{v}^{(1)} + \bar{\mathcal{V}}_i^{(1)} \right) + \mathcal{H} \bar{\nu}_i^{(1)} + D_i \bar{\Phi}^{(1)} \right\} \left(\bar{\nu}^i - D^i \bar{v}^{(1)} - \bar{\mathcal{V}}^i \right), \quad (\text{A}\cdot 41)$$

$$\begin{aligned}
 \mathcal{A}_{i\eta}^{(2)} = a \left\{ \mathcal{H} \left(\nu_i^{(2)} - D_i v^{(2)} - \mathcal{V}_i^{(2)} \right) - 2\mathcal{H} \Phi^{(1)} \nu_i^{(1)} \right. \\
 \left. + \left(\nu^j - D^j v^{(1)} - \mathcal{V}^j \right) \left(2D_i D_j v^{(1)} + 2D_i \mathcal{V}_j^{(1)} - 2\partial_\eta \Psi^{(1)} \gamma_{ij} \right. \right. \\
 \left. \left. + \partial_\eta \chi_{ij}^{(1)} - 2D_{(i} \nu_{j)}^{(1)} \right) \right\}, \quad (\text{A.42})
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{A}_{\eta i}^{(2)} = a \left\{ \left(\partial_\eta D_i v^{(2)} + \partial_\eta \mathcal{V}_i^{(2)} + D_i \Phi^{(2)} + \mathcal{H} \nu_i^{(2)} \right) - 2 \Phi^{(1)} \left(D_i \Phi^{(1)} + \mathcal{H} \nu_i^{(1)} \right) \right. \\
 \left. + \left(D^j v^{(1)} + \mathcal{V}^j - \nu^j \right) \left(2\partial_\eta \Psi^{(1)} \gamma_{ij} - \partial_\eta \chi_{ij}^{(1)} - 2D_{[i} \nu_{j]}^{(1)} \right) \right\}, \quad (\text{A.43})
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{A}_{ij}^{(2)} = a \left[D_i D_j v^{(2)} + D_i \mathcal{V}_j^{(2)} - D_{(i} \nu_{j)}^{(2)} - \left(2\mathcal{H} \Psi^{(2)} + \mathcal{H} \Phi^{(2)} + \partial_\eta \Psi^{(2)} \right) \gamma_{ij} \right. \\
 + \frac{1}{2} \partial_\eta \chi_{ij}^{(2)} + \mathcal{H} \chi_{ij}^{(2)} \\
 + \Phi^{(1)} \left\{ 2D_{(i} \nu_{j)}^{(1)} + \left(4\mathcal{H} \Psi^{(1)} + 3\mathcal{H} \Phi^{(1)} + 2\partial_\eta \Psi^{(1)} \right) \gamma_{ij} - \partial_\eta \chi_{ij}^{(1)} - 2\mathcal{H} \chi_{ij}^{(1)} \right\} \\
 + \left(D^k v^{(1)} + \mathcal{V}^k - \nu^k \right) \left(4D_{(i} \Psi^{(1)} \gamma_{j)k} - 2D_k \Psi^{(1)} \gamma_{ij} \right. \\
 \left. - 2D_{(i} \chi_{j)k}^{(1)} + D_k \chi_{ij}^{(1)} \right) \\
 \left. - \mathcal{H} \gamma_{ij} \nu_k^{(1)} \nu^k + \mathcal{H} \gamma_{ij} \left(D^k v^{(1)} + \mathcal{V}^k \right) \left(D_k v^{(1)} + \mathcal{V}_k^{(1)} \right) \right], \quad (\text{A.44})
 \end{aligned}$$

where we have used Eqs. (2.13), (2.15), (A.12)–(A.14), and the components of

$H_{ab}{}^c[\mathcal{H}]$ and $H_{ab}{}^c[\mathcal{L}]$ summarized in Appendix in KN2007. The components $\mathcal{A}_{\eta\eta}^{(1)}$, $\mathcal{A}_{i\eta}^{(1)}$, $\mathcal{A}_{\eta i}^{(2)}$, and $\mathcal{A}_{ij}^{(2)}$ are also derived from the perturbations of the identity $\bar{u}^b \bar{A}_{ab} = 0$.

A.3. Perturbation of the acceleration $\bar{a}_b = \bar{u}^c \bar{\nabla}_c \bar{u}_b$

Here, we consider the perturbations of the acceleration which is defined by

$$\bar{a}_b := \bar{u}^c \bar{\nabla}_c \bar{u}_b = \bar{u}^c \bar{A}_{cb}. \quad (\text{A.45})$$

To obtain the perturbations of the acceleration, it is convenient to introduce perturbations of the contravariant four-velocity \bar{u}^a which is expanded as

$$\bar{u}^a = u^a + \lambda (u^a)^{(1)} + \frac{1}{2} \lambda^2 (u^a)^{(2)} + O(\lambda^3), \quad (\text{A.46})$$

To obtain the relation between the perturbations \bar{u}^a and \bar{u}_a of the fluid four-velocity,

we have to consider the perturbations of the inverse metric \bar{g}^{ab} which is given by

$$\bar{g}^{ab} = g^{ab} - \lambda h^{ab} + \frac{1}{2} \lambda^2 \left(2h^{ac} h_c^b - l^{ab} \right) + O(\lambda^3). \quad (\text{A}\cdot 47)$$

Through Eqs. (2·25) and (A·47), we obtain the relation between perturbations of \bar{u}^a and \bar{u}_a of each order. Further, through Eqs. (2·8), (2·10), (2·26), and (2·28), the first- and the second-order perturbations of \bar{u}^a are also decomposed into the gauge-invariant and the gauge-variant parts as

$$\begin{pmatrix} (1) \\ u^a \end{pmatrix} = \begin{pmatrix} (1) \\ \mathcal{U}^a \end{pmatrix} - \mathcal{H}^{ab} u_b + \mathcal{L}_X u^a, \quad (\text{A}\cdot 48)$$

$$\begin{aligned} \begin{pmatrix} (2) \\ u^a \end{pmatrix} &= \begin{pmatrix} (2) \\ \mathcal{U}^a \end{pmatrix} - 2\mathcal{H}^{ab} \begin{pmatrix} (1) \\ u_b \end{pmatrix} + 2\mathcal{H}^{ac} \mathcal{H}_{cb} u^b - \mathcal{L}^{ab} u_b \\ &\quad + 2\mathcal{L}_X \left(g^{ab} \begin{pmatrix} (1) \\ u_b \end{pmatrix} - h^{ab} u_b \right) + \mathcal{L}_Y u^a - \mathcal{L}_X^2 u^a, \end{aligned} \quad (\text{A}\cdot 49)$$

where we used Eq. (3·7). These formulae are also given in KN2007 and we also note that Eqs. (A·48) and (A·49) have the same form as Eqs. (2·20) and (2·21), respectively.

Through the expansion formulae (A·28) and (A·46), the acceleration \bar{a}_b is also expanded as

$$\bar{a}_b = a_b + \lambda \begin{pmatrix} (1) \\ a_b \end{pmatrix} + \frac{1}{2} \lambda^2 \begin{pmatrix} (2) \\ a_b \end{pmatrix} + O(\lambda^3) \quad (\text{A}\cdot 50)$$

and we easily see that each order perturbation of the acceleration \bar{a}_b is given by

$$a_b := u^a A_{ab} = u^a \nabla_a u_b, \quad (\text{A}\cdot 51)$$

$$\begin{pmatrix} (1) \\ a_b \end{pmatrix} := u^a \begin{pmatrix} (1) \\ A_{ab} \end{pmatrix} + \begin{pmatrix} (1) \\ u^a \end{pmatrix} A_{ab}, \quad (\text{A}\cdot 52)$$

$$\begin{pmatrix} (2) \\ a_b \end{pmatrix} := u^a \begin{pmatrix} (2) \\ A_{ab} \end{pmatrix} + 2 \begin{pmatrix} (1) \\ u^a \end{pmatrix} \begin{pmatrix} (1) \\ A_{ab} \end{pmatrix} + \begin{pmatrix} (2) \\ u^a \end{pmatrix} A_{ab}. \quad (\text{A}\cdot 53)$$

Substituting Eqs. (A·32) and (A·48) into Eq. (A·52), we easily see that the first-order perturbation of the acceleration is decomposed into gauge-invariant and gauge-variant parts as

$$\begin{pmatrix} (1) \\ a_b \end{pmatrix} = \begin{pmatrix} (1) \\ \mathcal{A}_b \end{pmatrix} + \mathcal{L}_X a_b. \quad (\text{A}\cdot 54)$$

Further, substituting Eqs. (A·32), (A·34), (A·48), and (A·49) into Eq. (A·53), we easily see that the second-order perturbation of the acceleration is decomposed into gauge-invariant and gauge-variant parts as

$$\begin{pmatrix} (2) \\ a_b \end{pmatrix} = \begin{pmatrix} (2) \\ \mathcal{A}_b \end{pmatrix} + 2\mathcal{L}_X \begin{pmatrix} (1) \\ a_b \end{pmatrix} + \{ \mathcal{L}_Y - \mathcal{L}_X^2 \} a_b. \quad (\text{A}\cdot 55)$$

Here, we have defined the gauge-invariant parts $\begin{pmatrix} (1) \\ \mathcal{A}_b \end{pmatrix}$ and $\begin{pmatrix} (2) \\ \mathcal{A}_b \end{pmatrix}$ of the first- and the second-order perturbations of the accelerations by

$$\begin{pmatrix} (1) \\ \mathcal{A}_b \end{pmatrix} := u^a \begin{pmatrix} (1) \\ A_{ab} \end{pmatrix} + \begin{pmatrix} (1) \\ \mathcal{U}^a \end{pmatrix} A_{ab} - \mathcal{H}^{ac} u_c A_{ab}, \quad (\text{A}\cdot 56)$$

$$\begin{aligned}
 \mathcal{A}_b^{(2)} := & u^a \mathcal{A}_{ab}^{(2)} + \mathcal{U}^a A_{ab}^{(2)} - \mathcal{L}^{ac} u_c A_{ab} - 2\mathcal{H}^{ac} \mathcal{U}_c^{(1)} A_{ab} + 2\mathcal{H}^{ac} \mathcal{H}_{cd} u^d A_{ab} \\
 & + 2 \mathcal{A}_{ab}^{(1)} \mathcal{U}^a - 2 \mathcal{A}_{ab}^{(1)} \mathcal{H}^{ac} u_c.
 \end{aligned} \tag{A.57}$$

We note again that Eqs. (A.54) and (A.55) have the same forms as Eqs. (2.20) and (2.21), respectively.

Now, we consider the explicit components of the gauge-invariant parts of the perturbations of the acceleration \bar{a}_b associated with the fluid four-velocity \bar{u}_a through Eqs. (A.56) and (A.57). First, we consider the components of the background value of the acceleration associated with the four-velocity

$$a_b = u^a A_{ab} = a \mathcal{H} u^a \gamma_{ab} = 0, \tag{A.58}$$

where we have used Eq. (A.36). Next, though Eqs. (2.13), (A.12), (A.13), (A.36)–(A.40), and (A.56), the components of the gauge-invariant part $\mathcal{A}_a^{(1)}$ of the first-order perturbation of the acceleration are summarized as follows:

$$\mathcal{A}_\eta^{(1)} = 0, \tag{A.59}$$

$$\mathcal{A}_i^{(1)} = D_i \mathcal{P}^{(1)} + (\partial_\eta + \mathcal{H}) \left(D_i \mathcal{V}^{(1)} + \mathcal{V}_i^{(1)} \right). \tag{A.60}$$

Finally, we consider the components of the gauge-invariant part (A.57) of the second-order perturbation of the acceleration. By the direct calculations through Eqs. (2.13), (2.15), (3.7), (A.12)–(A.14), (A.36)–(A.40), and (A.41)–(A.44), the components of the gauge-invariant part $\mathcal{A}_a^{(2)}$ of the acceleration are given by

$$\mathcal{A}_\eta^{(2)} = 2 \mathcal{A}_i^{(1)} \left(\nu^i - D^i \mathcal{V}^{(1)} - \mathcal{V}^i \right), \tag{A.61}$$

$$\begin{aligned}
 \mathcal{A}_i^{(2)} = & (\partial_\eta + \mathcal{H}) \left(D_i \mathcal{V}^{(2)} + \mathcal{V}_i^{(2)} \right) + D_i \mathcal{P}^{(2)} \\
 & - 2 \mathcal{P}^{(1)} \left\{ 2 D_i \mathcal{P}^{(1)} + (\partial_\eta + \mathcal{H}) \left(D_i \mathcal{V}^{(1)} + \mathcal{V}_i^{(1)} \right) \right\} \\
 & + 2 \left(\nu^j - D^j \mathcal{V}^{(1)} - \mathcal{V}^j \right) \left(D_i \mathcal{V}_j^{(1)} - D_j D_i \mathcal{V}^{(1)} - D_j \mathcal{V}_i^{(1)} \right).
 \end{aligned} \tag{A.62}$$

We can easily check the components (A.59)–(A.62) are consistent with the perturbation of the identity $\bar{u}^b \bar{a}_b = 0$.

A.4. Perturbation of the tensor $\bar{B}_{ab} := \bar{A}_{ab} + \bar{u}_a \bar{a}_b$

Here, we consider the perturbations of the tensor field \bar{B}_{ab} defined by Eq. (A.2), which is also written by

$$\bar{B}_{ab} = \bar{A}_{ab} + \bar{u}_a \bar{a}_b \tag{A.63}$$

through the tensor \bar{A}_{ab} defined by Eq. (A·22). The tensor \bar{B}_{ab} is also decomposed into the trace part and the traceless part. Further, the traceless part of the tensor \bar{B}_{ab} is also classified into the symmetric part and the antisymmetric part:

$$\bar{B}_{ab} = \frac{1}{3}\bar{q}_{ab}\bar{\theta} + \bar{\sigma}_{ab} + \bar{\omega}_{ab}, \quad (\text{A}\cdot 64)$$

$$\bar{\theta} := \bar{q}^{ab}\bar{B}_{ab} = \bar{\nabla}_c \bar{u}^c, \quad (\text{A}\cdot 65)$$

$$\bar{\sigma}_{ab} := \bar{B}_{(ab)} - \frac{1}{3}\bar{q}_{ab}\bar{\theta}, \quad (\text{A}\cdot 66)$$

$$\bar{\omega}_{ab} := \bar{B}_{[ab]}. \quad (\text{A}\cdot 67)$$

$\bar{\theta}$, $\bar{\sigma}_{ab}$, and $\bar{\omega}_{ab}$ are called the expansion, the shear, and the rotation associated with the four-velocity \bar{u}_a , respectively.

The perturbative expansions of the tensor \bar{B}_{ab} is given by

$$\bar{B}_{ab} = B_{ab} + \lambda \overset{(1)}{B}_{ab} + \frac{1}{2}\lambda^2 \overset{(2)}{B}_{ab} + O(\lambda^3). \quad (\text{A}\cdot 68)$$

On the other hand, the perturbative expansion of the tensor \bar{B}_{ab} is also given by the direct expansion of Eq. (A·63). Comparing these perturbative expansions, we obtain

$$B_{ab} = A_{ab} + u_a a_b, \quad (\text{A}\cdot 69)$$

$$\overset{(1)}{B}_{ab} = \overset{(1)}{A}_{ab} + u_a \overset{(1)}{a}_b + (\overset{(1)}{u}_a) \overset{(1)}{a}_b, \quad (\text{A}\cdot 70)$$

$$\overset{(2)}{B}_{ab} = \overset{(2)}{A}_{ab} + u_a \overset{(2)}{a}_b + (\overset{(2)}{u}_a) \overset{(2)}{a}_b + 2 (\overset{(1)}{u}_a) \overset{(1)}{a}_b. \quad (\text{A}\cdot 71)$$

Further, substituting Eqs. (A·32), (A·54), and (2·28) into Eq. (A·70), we can decompose the first-order perturbation $\overset{(1)}{B}_{ab}$ of \bar{B}_{ab} into gauge-invariant and gauge-variant parts as

$$\overset{(1)}{B}_{ab} = \overset{(1)}{\mathcal{B}}_{ab} + \mathcal{L}_X \overset{(1)}{B}_{ab}, \quad (\text{A}\cdot 72)$$

where the gauge-invariant part $\overset{(1)}{\mathcal{B}}_{ab}$ is defined by

$$\overset{(1)}{\mathcal{B}}_{ab} := \overset{(1)}{\mathcal{A}}_{ab} + u_a \overset{(1)}{\mathcal{A}}_b + \overset{(1)}{\mathcal{U}}_a \overset{(1)}{a}_b. \quad (\text{A}\cdot 73)$$

Note that Eq. (A·72) has the same form as the decomposition formula (2·20). Further, substituting Eqs. (2·26), (2·28), (A·34), (A·54), (A·55), the last equation in Eqs. (2·26) and (2·28) into Eq. (A·71), the second-order perturbation $\overset{(2)}{B}_{ab}$ of the tensor \bar{B}_{ab} is decomposed as

$$\overset{(2)}{B}_{ab} = \overset{(2)}{\mathcal{B}}_{ab} + 2\mathcal{L}_X \overset{(1)}{B}_{ab} + (\mathcal{L}_Y - \mathcal{L}_X^2) \overset{(1)}{B}_{ab}, \quad (\text{A}\cdot 74)$$

where the gauge-invariant part $\overset{(2)}{\mathcal{B}}_{ab}$ is defined as

$$\overset{(2)}{\mathcal{B}}_{ab} := \overset{(2)}{\mathcal{A}}_{ab} + u_a \overset{(2)}{\mathcal{A}}_b + \overset{(2)}{\mathcal{U}}_a \overset{(2)}{a}_b + 2 \overset{(1)}{\mathcal{U}}_a \overset{(1)}{a}_b. \quad (\text{A}\cdot 75)$$

We also note that Eq. (A.74) has the same form as Eq. (2.21).

Now, we consider the components of the gauge-invariant parts $\mathcal{B}_{ab}^{(1)}$ and $\mathcal{B}_{ab}^{(2)}$ of the first- and the second-order perturbations of the tensor field \bar{B}_{ab} . First, we note that the background value of the tensor field \bar{B}_{ab} is given by Eq. (A.36), (A.58), and (A.69):

$$B_{ab} = a\mathcal{H}\gamma_{ab}. \quad (\text{A.76})$$

Through Eqs. (A.12), (A.13), (A.58)–(A.60), and (A.73), the components of the gauge-invariant part $\mathcal{B}_{ab}^{(1)}$ of the first-order perturbation of the tensor \bar{B}_{ab} are summarized as

$$\mathcal{B}_{\eta\eta}^{(1)} = 0, \quad (\text{A.77})$$

$$\mathcal{B}_{\eta i}^{(1)} = a\mathcal{H} \left(\nu_i^{(1)} - D_i v^{(1)} - \mathcal{V}_i^{(1)} \right) = \mathcal{B}_{i\eta}^{(1)}, \quad (\text{A.78})$$

$$\begin{aligned} \mathcal{B}_{ij}^{(1)} = a \left\{ D_i \left(D_j v^{(1)} + \mathcal{V}_j^{(1)} \right) - D_{(i} \nu_{j)}^{(1)} - \left(\mathcal{H} \Phi^{(1)} + 2\mathcal{H} \Psi^{(1)} + \partial_\eta \Psi^{(1)} \right) \gamma_{ij} \right. \\ \left. + \frac{1}{2} (\partial_\eta + 2\mathcal{H}) \chi_{ij}^{(1)} \right\}. \end{aligned} \quad (\text{A.79})$$

Finally, through Eqs. (A.12)–(A.15), (A.41)–(A.44), (A.59)–(A.62), (A.75), the components of the gauge-invariant part $\mathcal{B}_{ab}^{(2)}$ of the second-order perturbation of the tensor \bar{B}_{ab} are summarized as follows:

$$\mathcal{B}_{\eta\eta}^{(2)} = 2a\mathcal{H} \left(D_i v^{(1)} + \mathcal{V}_i^{(1)} - \nu_i^{(1)} \right) \left(D^i v^{(1)} + \mathcal{V}^i^{(1)} - \nu^i^{(1)} \right), \quad (\text{A.80})$$

$$\begin{aligned} \mathcal{B}_{\eta i}^{(2)} = a \left[2 \left(\nu_j^{(1)} - D^j v^{(1)} - \mathcal{V}_j^{(1)} \right) \left\{ D_j \left(D_i v^{(1)} + \mathcal{V}_i^{(1)} \right) - D_{(i} \nu_{j)}^{(1)} - \partial_\eta \Psi^{(1)} \gamma_{ij} + \frac{1}{2} \partial_\eta \chi_{ij}^{(1)} \right\} \right. \\ \left. + \mathcal{H} \left(\nu_i^{(2)} - D_i v^{(2)} - \mathcal{V}_i^{(2)} \right) - 2\mathcal{H} \Phi^{(1)} \nu_i^{(1)} \right], \end{aligned} \quad (\text{A.81})$$

$$\begin{aligned} \mathcal{B}_{i\eta}^{(2)} = a \left[2 \left(\nu_j^{(1)} - D^j v^{(1)} - \mathcal{V}_j^{(1)} \right) \left\{ D_i \left(D_j v^{(1)} + \mathcal{V}_j^{(1)} \right) - D_{(i} \nu_{j)}^{(1)} - \partial_\eta \Psi^{(1)} \gamma_{ij} + \frac{1}{2} \partial_\eta \chi_{ij}^{(1)} \right\} \right. \\ \left. + \mathcal{H} \left(\nu_i^{(2)} - D_i v^{(2)} - \mathcal{V}_i^{(2)} \right) - 2\mathcal{H} \Phi^{(1)} \nu_i^{(1)} \right], \end{aligned} \quad (\text{A.82})$$

$$\begin{aligned} \mathcal{B}_{ij}^{(2)} = a \left[D_i \left(D_j v^{(2)} + \mathcal{V}_j^{(2)} \right) - D_{(i} \nu_{j)}^{(2)} - \left\{ \mathcal{H} \left(2 \Psi^{(2)} + \Phi^{(2)} \right) + \partial_\eta \Psi^{(2)} \right\} \gamma_{ij} \right. \\ \left. + \frac{1}{2} (\partial_\eta + 2\mathcal{H}) \chi_{ij}^{(2)} \right] \end{aligned}$$

$$\begin{aligned}
& + \overset{(1)}{\Phi} \left\{ 2D_{(i} \overset{(1)}{\nu}_{j)} + \left(2\partial_\eta \overset{(1)}{\Psi} + \mathcal{H} \left(4 \overset{(1)}{\Psi} + 3 \overset{(1)}{\Phi} \right) \right) \gamma_{ij} - (\partial_\eta + 2\mathcal{H}) \overset{(1)}{\chi}_{ij} \right\} \\
& + 2 \left(D_i \overset{(1)}{v} + \overset{(1)}{\mathcal{V}}_i \right) \left\{ D_j \overset{(1)}{\Phi} + (\partial_\eta + \mathcal{H}) \left(D_j \overset{(1)}{v} + \overset{(1)}{\mathcal{V}}_j \right) \right\} \\
& + \left(\overset{(1)}{\nu}^k - D^k \overset{(1)}{v} - \overset{(1)}{\mathcal{V}}^k \right) \left(2D_k \overset{(1)}{\Psi} \gamma_{ij} - 4D_{(i} \overset{(1)}{\Psi} \gamma_{j)k} + 2D_{(i} \overset{(1)}{\chi}_{j)k} - D_k \overset{(1)}{\chi}_{ij} \right) \\
& - \mathcal{H} \gamma_{ij} \overset{(1)}{\nu}_k \overset{(1)}{\nu}^k + \mathcal{H} \gamma_{ij} \left(D^k \overset{(1)}{v} + \overset{(1)}{\mathcal{V}}^k \right) \left(D_k \overset{(1)}{v} + \overset{(1)}{\mathcal{V}}_k \right) \Big]. \tag{A.83}
\end{aligned}$$

We also note that the components $\overset{(1)}{\mathcal{B}}_{\eta\eta}$, $\overset{(1)}{\mathcal{B}}_{i\eta}$, $\overset{(1)}{\mathcal{B}}_{\eta i}$ for the first-order perturbation $\overset{(1)}{\mathcal{B}}_{ab}$ and the components $\overset{(2)}{\mathcal{B}}_{\eta\eta}$, $\overset{(2)}{\mathcal{B}}_{i\eta}$, $\overset{(2)}{\mathcal{B}}_{\eta i}$ for the second-order perturbation $\overset{(2)}{\mathcal{B}}_{ab}$ are also derived from the perturbations of the properties

$$\bar{u}^a \bar{B}_{ab} = \bar{u}^a \bar{B}_{ba} = 0. \tag{A.84}$$

A.4.1. Expansion $\bar{\theta}$

Now, we consider the perturbation of the expansion $\bar{\theta}$ defined by Eq. (A.65). Through Eqs. (A.47) and (A.68), the expansion $\bar{\theta}$ is expanded as

$$\bar{\theta} = \theta + \lambda \overset{(1)}{\theta} + \frac{1}{2} \lambda^2 \overset{(2)}{\theta} + O(\lambda^3), \tag{A.85}$$

where each order perturbations of $\bar{\theta}$ is given by

$$\theta = g^{ab} B_{ab} = g^{ab} A_{ab} = \nabla_a u^a, \tag{A.86}$$

$$\overset{(1)}{\theta} = g^{ab} \overset{(1)}{B}_{ab} - B_{ab} h^{ab}, \tag{A.87}$$

$$\overset{(2)}{\theta} = g^{ab} \overset{(2)}{B}_{ab} - 2 \overset{(1)}{B}_{ab} h^{ab} + B_{ab} (2h^{ae} h_e{}^b - l^{ab}). \tag{A.88}$$

Through Eqs. (2.8) and (A.72), the first-order perturbation $\overset{(1)}{\theta}$ is decomposed as

$$\overset{(1)}{\theta} =: \overset{(1)}{\Theta} + \mathcal{L}_X \theta, \tag{A.89}$$

where the gauge-invariant part $\overset{(1)}{\Theta}$ of $\overset{(1)}{\theta}$ is given by

$$\overset{(1)}{\Theta} =: g^{ab} \overset{(1)}{\mathcal{B}}_{ab} - B_{ab} \mathcal{H}^{ab}. \tag{A.90}$$

Similarly, through Eqs. (2.8), (2.10), (A.72), and (A.74), the second-order perturbation $\overset{(2)}{\theta}$ is also decomposed as

$$\overset{(2)}{\theta} =: \overset{(2)}{\Theta} + 2\mathcal{L}_X \overset{(1)}{\theta} + \{ \mathcal{L}_Y - \mathcal{L}_X^2 \} \theta, \tag{A.91}$$

where the gauge-invariant part $\Theta^{(2)}$ of $\theta^{(2)}$ is given by

$$\Theta^{(2)} := g^{ab} \mathcal{B}_{ab}^{(2)} - B_{ab} \mathcal{L}^{ab} - 2 \mathcal{B}_{ab}^{(1)} \mathcal{H}^{ab} + 2 B_{ab} \mathcal{H}^{ae} \mathcal{H}_e{}^b. \quad (\text{A}\cdot 92)$$

Here again, we note that the decompositions (A·89) and (A·91) of the perturbations of the expansion $\bar{\theta}$ into the gauge-invariant and the gauge-variant parts have the same forms as Eqs. (2·20) and (2·21), respectively.

Now, we derive the explicit expression of the perturbations of the expansion $\bar{\theta}$. First, we consider the background value of the expansion (A·86). From Eq. (A·76), the background value θ of $\bar{\theta}$ is given by

$$\theta = 3 \frac{1}{a} \mathcal{H} = 3 \frac{\partial_\eta a}{a^2} = 3 \frac{1}{a} \frac{da}{d\tau} = 3H, \quad (\text{A}\cdot 93)$$

where H is the Hubble parameter of the background universe and $d\tau = a d\eta$. Second, through Eqs. (2·13), and (A·76)–(A·79), the gauge-invariant part (A·90) of the first-order perturbation $\theta^{(1)}$ is given by

$$\Theta^{(1)} = \frac{1}{a} \left(\Delta^{(1)} v - 3\mathcal{H} \Phi^{(1)} - 3\partial_\eta \Psi^{(1)} \right). \quad (\text{A}\cdot 94)$$

Finally, through Eqs. (2·13), (2·15), and (A·76)–(A·83), the gauge-invariant part (A·92) of the second-order perturbation of the expansion is given by

$$\begin{aligned} \Theta^{(2)} = \frac{1}{a} & \left[\Delta^{(2)} v - 3\partial_\eta \Psi^{(2)} - 3\mathcal{H} \Phi^{(2)} + 6\partial_\eta \Psi^{(1)} \left(\Phi^{(1)} - 2\Psi^{(1)} \right) \right. \\ & + 4 \Psi^{(1)} \Delta^{(1)} v + 9\mathcal{H} \left(\Phi^{(1)} \right)^2 + 2 \nu^k D_k \Psi^{(1)} - 3\mathcal{H} \nu^k \nu_k^{(1)} \\ & + \left(D^k v^{(1)} + \mathcal{V}^k \right) \left\{ 2D_k \left(\Phi^{(1)} - \Psi^{(1)} \right) + (2\partial_\eta + 3\mathcal{H}) \left(D_k v^{(1)} + \mathcal{V}_k^{(1)} \right) \right\} \\ & \left. + \chi^{ik} \left(2D_i \nu_k^{(1)} - 2D_i D_k v^{(1)} - 2D_i \mathcal{V}_k^{(1)} - \partial_\eta \chi_{ik}^{(1)} \right) \right]. \quad (\text{A}\cdot 95) \end{aligned}$$

A.4.2. Shear $\bar{\sigma}_{ab}$

The perturbations of the shear tensor $\bar{\sigma}_{ab}$ defined by Eq. (A·66) are given as follows. Through Eqs. (A·4), (A·68), and (A·85), the perturbative expansion of the shear tensor $\bar{\sigma}_{ab}$ is given by

$$\bar{\sigma}_{ab} =: \sigma_{ab} + \lambda (\sigma_{ab})^{(1)} + \frac{1}{2} \lambda^2 (\sigma_{ab})^{(2)} + O(\lambda^3), \quad (\text{A}\cdot 96)$$

where

$$\sigma_{ab} := B_{(ab)} - \frac{1}{3} q_{ab} \theta, \quad (\text{A}\cdot 97)$$

$${}^{(1)}(\sigma_{ab}) := B_{(ab)}^{(1)} - \frac{1}{3} {}^{(1)}(q_{ab}) \theta - \frac{1}{3} q_{ab} {}^{(1)}\theta, \quad (\text{A}\cdot 98)$$

$${}^{(2)}(\sigma_{ab}) := B_{(ab)}^{(2)} - \frac{1}{3} q_{ab} {}^{(2)}\theta - \frac{2}{3} {}^{(1)}(q_{ab}) {}^{(1)}\theta - \frac{1}{3} {}^{(2)}(q_{ab}) \theta. \quad (\text{A}\cdot 99)$$

Through Eqs. (A·8), (A·72), and (A·89), the first-order perturbation (A·98) of the shear tensor is decomposed as

$${}^{(1)}(\sigma_{ab}) = \Sigma_{ab}^{(1)} + \mathcal{L}_X \sigma_{ab}, \quad (\text{A}\cdot 100)$$

where

$$\Sigma_{ab}^{(1)} := \mathcal{B}_{(ab)}^{(1)} - \frac{1}{3} \mathcal{Q}_{ab}^{(1)} \theta - \frac{1}{3} q_{ab} {}^{(1)}\Theta. \quad (\text{A}\cdot 101)$$

Similarly, through Eqs. (A·8), (A·9), (A·72), (A·74), (A·89), and (A·91), the second-order perturbation (A·99) of the shear tensor is decomposed as

$${}^{(2)}\sigma_{ab} = \Sigma_{ab}^{(2)} + 2\mathcal{L}_X {}^{(1)}(\sigma_{ab}) + \mathcal{L}_Y \sigma_{ab} - \mathcal{L}_X^2 \sigma_{ab}, \quad (\text{A}\cdot 102)$$

where

$$\Sigma_{ab}^{(2)} := \mathcal{B}_{(ab)}^{(2)} - \frac{1}{3} q_{ab} {}^{(2)}\Theta - \frac{1}{3} \mathcal{Q}_{ab}^{(2)} \theta - \frac{2}{3} {}^{(1)}\mathcal{Q}_{ab} {}^{(1)}\Theta. \quad (\text{A}\cdot 103)$$

Here again, Eqs. (A·100) and (A·102) show that the perturbations of the shear tensor are decomposed into the gauge-invariant and the gauge-variant parts in the same form as Eqs. (2·20) and (2·21), respectively.

Now, we consider the explicit components of the shear tensor of each order perturbations. First, from Eqs. (A·16), (A·76), (A·93), and (A·97), the background shear tensor is given by

$$\sigma_{ab} = 0. \quad (\text{A}\cdot 104)$$

Second, through Eqs. (A·16)–(A·18), (A·77)–(A·79), and (A·94), the components of the gauge-invariant part $\Sigma_{ab}^{(1)}$ of the first-order perturbation of the shear tensor are given by

$$\Sigma_{\eta\eta}^{(1)} = \Sigma_{\eta i}^{(1)} = \Sigma_{i\eta}^{(1)} = 0, \quad (\text{A}\cdot 105)$$

$$\Sigma_{ij}^{(1)} = a \left\{ \left(D_i D_j - \frac{1}{3} \gamma_{ij} \Delta \right) {}^{(1)}v + D_{(i} {}^{(1)}\mathcal{V}_{j)} - D_{(i} {}^{(1)}\mathcal{V}_{j)} + \frac{1}{2} \partial_\eta {}^{(1)}\chi_{ij} \right\}. \quad (\text{A}\cdot 106)$$

Finally, through Eqs. (A·16)–(A·21), (A·80)–(A·83), and (A·93)–(A·95), the components of the gauge-invariant part $\Sigma_{ab}^{(2)}$ of the second-order perturbation of the shear

tensor are summarized as

$$\Sigma_{\eta\eta}^{(2)} = 0, \quad (\text{A}\cdot 107)$$

$$\Sigma_{i\eta}^{(2)} = \Sigma_{\eta i}^{(2)} = 2 \left(\nu^{(1)j} - D^{(1)j} v - \mathcal{V}^{(1)j} \right) \Sigma_{ij}^{(1)}, \quad (\text{A}\cdot 108)$$

$$\begin{aligned} \Sigma_{ij}^{(2)} = a & \left[\left(D_i D_j - \frac{1}{3} \gamma_{ij} \Delta \right) \begin{pmatrix} (2) \\ v \end{pmatrix} + D_{(i} \begin{pmatrix} (2) \\ \mathcal{V}_{j)} \end{pmatrix} - D_{(i} \begin{pmatrix} (2) \\ \nu_{j)} \end{pmatrix} + \frac{1}{2} \partial_\eta \begin{pmatrix} (2) \\ \chi_{ij} \end{pmatrix} \right. \\ & + \Phi^{(1)} \left(2 D_{(i} \begin{pmatrix} (1) \\ \nu_{j)} \end{pmatrix} - \partial_\eta \begin{pmatrix} (1) \\ \chi_{ij} \end{pmatrix} \right) \\ & + 2 \left(D^k \begin{pmatrix} (1) \\ v \end{pmatrix} + \mathcal{V}^{(1)k} \right) \left\{ \left(\gamma_{k(i} D_{j)} - \frac{1}{3} \gamma_{ij} D_k \right) \left(\Phi^{(1)} + 2 \Psi^{(1)} \right) \right. \\ & \quad + \partial_\eta \left(\gamma_{k(i} D_{j)} \begin{pmatrix} (1) \\ v \end{pmatrix} + \gamma_{k(i} \begin{pmatrix} (1) \\ \mathcal{V}_{j)} \end{pmatrix} \right) \\ & \quad - \frac{1}{3} \gamma_{ij} \partial_\eta \left(D_k \begin{pmatrix} (1) \\ v \end{pmatrix} + \mathcal{V}_k^{(1)} \right) \\ & \quad \left. \left. - D_{(i} \begin{pmatrix} (1) \\ \chi_{j)k} \right) + \frac{1}{2} D_k \begin{pmatrix} (1) \\ \chi_{ij} \end{pmatrix} \right\} \right. \\ & + 2 \nu^{(1)k} \left\{ -2 \left(\gamma_{k(i} D_{j)} - \frac{1}{3} \gamma_{ij} D_k \right) \begin{pmatrix} (1) \\ \Psi \end{pmatrix} + D_{(i} \begin{pmatrix} (1) \\ \chi_{j)k} \right) - \frac{1}{2} D_k \begin{pmatrix} (1) \\ \chi_{ij} \end{pmatrix} \right\} \\ & + \frac{2}{3} \chi^{(1)lk} \left\{ -\gamma_{ij} D_l \begin{pmatrix} (1) \\ \nu_k \end{pmatrix} + \gamma_{ij} D_l \left(D_k \begin{pmatrix} (1) \\ v \end{pmatrix} + \mathcal{V}_k^{(1)} \right) + \frac{1}{2} \gamma_{ij} \partial_\eta \begin{pmatrix} (1) \\ \chi_{lk} \end{pmatrix} \right. \\ & \quad \left. \left. - \gamma_{ik} \gamma_{jl} \left(\Delta \begin{pmatrix} (1) \\ v \end{pmatrix} - \frac{1}{3} \partial_\eta \begin{pmatrix} (1) \\ \Psi \end{pmatrix} \right) \right\} \right]. \quad (\text{A}\cdot 109) \end{aligned}$$

A.4.3. Rotation $\bar{\omega}_{ab}$

The perturbations of the rotation defined by Eq. (A·67) are given by as follows. The perturbative expansion of the rotation is directly derived from the perturbative expansion (A·68) of the tensor field \bar{B}_{ab} and $\bar{\omega}_{ab}$ is expanded as

$$\bar{\omega}_{ab} =: \omega_{ab} + \lambda \omega_{ab}^{(1)} + \frac{1}{2} \lambda^2 \omega_{ab}^{(2)} + O(\lambda^3), \quad (\text{A}\cdot 110)$$

and we have

$$\omega_{ab} := B_{[ab]}, \quad \omega_{ab}^{(1)} := (B_{[ab]}^{(1)}), \quad \omega_{ab}^{(2)} := (B_{[ab]}^{(2)}). \quad (\text{A}\cdot 111)$$

We can also define the gauge-invariant expression of the first- and the second-order perturbation of the rotation:

$$\omega_{ab}^{(1)} = \Omega_{ab}^{(1)} + \mathcal{L}_X \omega_{ab}, \quad \omega_{ab}^{(2)} = \Omega_{ab}^{(2)} + 2 \mathcal{L}_X \omega_{ab}^{(1)} + (\mathcal{L}_Y - \mathcal{L}_X^2) \omega_{ab}, \quad (\text{A}\cdot 112)$$

where the gauge-invariant variables for the first- and the second-order perturbations of the rotation are given by

$$\Omega_{ab}^{(1)} = \mathcal{B}_{[ab]}^{(1)}, \quad \Omega_{ab}^{(2)} = \mathcal{B}_{[ab]}^{(2)}. \quad (\text{A}\cdot 113)$$

From Eq. (A.76), the background value of the rotation is given by

$$\omega_{ab} = a\mathcal{H}\gamma_{[ab]} = 0. \quad (\text{A}\cdot 114)$$

Through Eqs. (A.77)–(A.79), the components of the gauge-invariant part $\Omega_{ab}^{(1)}$ of the first-order perturbation of the rotation are given by

$$\Omega_{\eta\eta}^{(1)} = \Omega_{\eta i}^{(1)} = \Omega_{i\eta}^{(1)} = 0, \quad \Omega_{ij}^{(1)} = aD_{[i} \mathcal{V}_{j]}^{(1)}. \quad (\text{A}\cdot 115)$$

Further, through Eqs. (A.80)–(A.83), the components of the gauge-invariant part of the second-order perturbation of the rotation are given by

$$\Omega_{\eta\eta}^{(2)} = 0, \quad (\text{A}\cdot 116)$$

$$\Omega_{i\eta}^{(2)} = -\Omega_{\eta i}^{(2)} = 2a \left(\nu^{(1)j} - D^j v^{(1)} - \mathcal{V}^{(1)j} \right) \left(D_{[i} \mathcal{V}_{j]}^{(1)} \right), \quad (\text{A}\cdot 117)$$

$$\Omega_{ij}^{(2)} = a \left\{ D_{[i} \mathcal{V}_{j]}^{(2)} - 2 \left(D^k v^{(1)} + \mathcal{V}^{(1)k} \right) \left(D_{[i} \mathcal{V}_{j]}^{(1)} + \partial_{\eta} D_{[i} v^{(1)} \gamma_{j]k} + \partial_{\eta} \mathcal{V}_{[i}^{(1)} \gamma_{j]k} \right) \right\}. \quad (\text{A}\cdot 118)$$

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